Appendix A. Longwave radiation in a canopy stand

The longwave radiation transfer equations can be written as (Ross, 1981):

$$\frac{di_{ld}(L,r)}{dL}\cos\theta = -G(L,r)i_{ld}(L,r) + \eta(L,r) \qquad \text{(A.1)}$$

$$\frac{di_{lu}(L,r)}{dL}\cos\phi = G(L,r)i_{lu}(L,r) - \eta(L,r) \qquad \text{(A.2)}$$

where $i_{ld}(L,r)$, $i_{lu}(L,r)$ are the longwave downward and upward radiation intensities inside a plant stand, r is the direction in space, θ is the inclination angle and ϕ is equal to $(\pi-\theta)$, $G(L,r)$ is the foliage area orientation function, $\eta(L,r)$ is the leaf emission coefficient.

The boundary conditions are given as:

$$\begin{aligned} i_{ld}(0) &= \varepsilon_A \\ i_{lu}(L_0) &= \varepsilon_B + \frac{A_{BJ}}{\pi}\int_{\Omega_1} i_d(L_0, r')\cos\theta' \, d\,\Omega' \end{aligned} \qquad \text{(A.3)}$$

where ε_A is the intensity of longwave radiation of the atmosphere, ε_B is that emitted from the ground surface below the stand, A_{BJ} is the albedo of the ground surface for longwave radiation.

By assuming the leaf temperature is the same in a horizontal layer, one can obtain the following expressions:

$$\begin{aligned} R_{ld}(L) = {} & 2\pi\int_0^{\pi/2}\varepsilon_A(\theta)\exp\left[-\frac{LG(\theta)}{\cos\theta}\right]\cos\theta\sin\theta\, d\theta + \\ & \int_0^L R_{lc}(L')\left\{2\int_0^{\pi/2} G(\theta)\exp\left[-(L-L')\frac{G(\theta)}{\cos\theta}\right]\sin\theta d\theta\right\}dL' \end{aligned} \qquad \text{(A.4)}$$

$$R_{lu}(L) = 2R_{lg}\int_0^{\pi/2} exp\left[-(L_0 - L)\frac{G(\phi)}{cos\,\phi}\right]cos\,\phi\, sin\,\phi\, d\phi +$$
$$\int_L^{L_0} R_{lc}(L')\left\{2\int_0^{\pi/2} G(\phi) exp\left[-(L_0 - L)\frac{G(\phi)}{cos\,\phi}\right]cos\,\phi\, sin\,\phi d\phi\right\}dL' +$$
$$2\pi(1-\delta_B)\int_0^{\pi/2} exp\left[-(L_0 - L)\frac{G(\phi)}{cos\,\phi}\right]cos\,\phi\, sin\,\phi d\phi \times$$
$$2\pi\int_0^{\pi/2}\varepsilon_A(\theta) exp\left[-L_0\frac{G(\theta)}{cos\theta}\right]cos\theta sin\theta d\theta + \qquad (A.5)$$
$$2\pi(1-\delta_B)\int_0^{\pi/2} exp\left[-(L_0 - L)\frac{G(\phi)}{cos\,\phi}\right]cos\,\phi\, sin\,\phi\, d\phi \times$$
$$\int_0^{L_0} E_L(L)\left\{2\int_0^{\pi/2} G(\theta) exp\left[-(L_0 - L)\frac{G(\theta)}{cos\,\theta}\right]sin\,\theta d\theta\right\}dL'$$

where $R_{ld}(L)$ and $R_{lu}(L)$ are the longwave downward and upward irradiances.

The first term in equation (A.4) defines that part of longave radiation of the atmosphere which penetrates the layer (0, L) of the stand and reaches the level L, the second term defines the downward flux of the longwave radiation of leaves in the layer (0, L), which reaches the level L. In equation (A.5), the first term defines that part of longwave radiation of the ground which penetrates the layer (L, L_0) and reaches the level L, the second term describes the upward flux of longwave radiation of leaves in the layer (L, L_0) which reaches the level L, and the third term defines that part of longwave radiation of the atmosphere which penetrates the whole layer of the stand, is reflected from the ground and, penetrating through the layer (L, L_0), reaches the level L. The last term represents that part of the downward flux of radiation of the leaves off the whole stand which reaches the ground, is reflected from it, penetrates the layer (L, L_0) and reaches the level L.

Since $(1-\delta_B) \approx 0.05$, the last two terms in expression (A.5) can be neglected as a first approximation and in the case of isotropic longwave radiation of the atmosphere, we can obtain the following expressions from equations (A.4) and (A.5):

$$R_{ld}(L) = R_{lc} - (R_{lc} - R_{la})a_D(L) \qquad (A.6)$$

$$R_{lu}(L) = R_{lc} - (R_{lc} - R_{lg})a_D(L) \qquad (A.7)$$

where R_{lc} is the longwave radiation emitted from the stand, R_{la} is the longwave radiation from the atmosphere, R_{lc} is the longwave radiation from ground surface, $a_D(L)$ is the penetration function

$$a_D(L) = 2\int_0^{\pi/2} exp(-\frac{LG(\theta)}{cos\theta})cos\,\theta\, sin\theta\, d\theta \qquad (A.8)$$

The net longwave radiation can be written as:

$$R_{ln}(L) = -(R_{lc} - R_{la})a_D(L) - (R_{lg} - R_{lc})a_D(L_0 - L) \qquad \text{(A.9)}$$

For relatively dense stand, $R_{lg} \approx R_{lc}$, $R_{lu} = R_{lc}$ and one obtains:

$$R_{ln} = (R_{la} - R_{lc})a_D(L) \qquad \text{(A.10)}$$

The penetration function for longwave radiation can be defined as:

$$a_D(L) = 2\int_0^{\pi/2} exp(-\frac{LG(\theta)}{cos\,\theta})\,cos\,\theta\,sin\theta\,d\theta \qquad \text{(A.11)}$$

where L is the leaf area index, *G(θ)* is the so-called G-function which is a complicated double integral of inclination (θ) and azimuth orientation. This equation can be equally applied to diffuse radiation in a plant canopy.

The penetration function for shortwave radiation in a canopy with uniform orientation (spherical orientation) can be expressed as:

$$a_s(L) = exp(\frac{-L}{2sin\,\beta}) \qquad \text{(A.12)}$$

where β is the solar elevation.

It is clear that both penetration functions are exponential. Data from Ross (1981, Table II.5.1) presented in Figure A1 shows that both shortwave and longwave radiation can be approximated by the same function (Beer's law) with the same parameter value (bulk extinction coefficient).

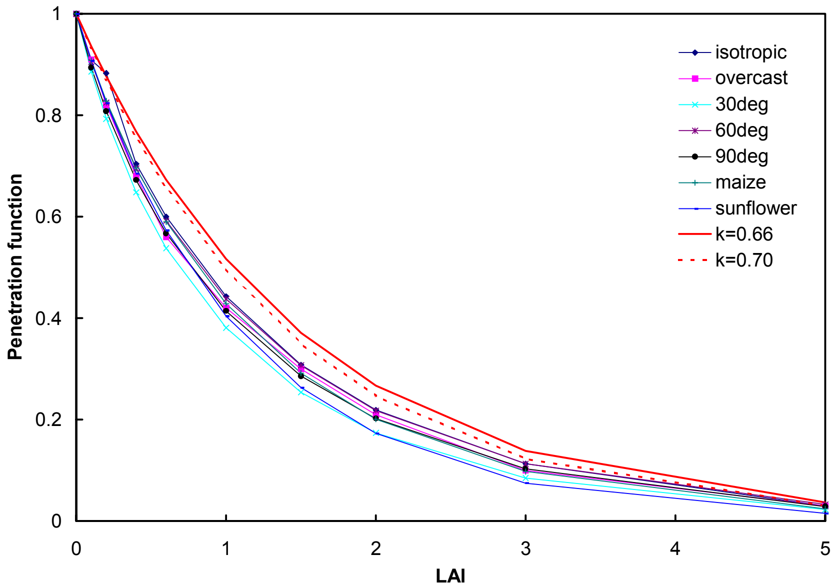

Figure A1. Penetration function for shortwave and longwave radiation in a spherical canopy under various sky conditions.

Appendix B: Details of Analytical Solutions

We here formulate u and u' (i.e. $\partial u/\partial\zeta$) for the analytical solutions presented in section 4.9.4, equations (4.13) and (4.14). The most general formulation (*Broadbridge*, 1990) has not been given succinctly, or in a form that permits exact evaluation.

We express u as:

$$u = u_0 + u_{s1} + u_{s2} \tag{B.1}$$

where u_0 is required regardless of initial and lower boundary conditions, u_{s1} and u_{s2} are non-zero summations when there is a finite-depth profile, and u_{s2} is non-zero only for a finite-depth profile with non-zero initial soil-water content.

To represent the effect of non-zero initial soil-water content $\Theta_i^\ddagger$, we define:

$$Q = \rho + 0.5\frac{\Theta_i^\ddagger}{1-\Theta_i^\ddagger} \tag{B.2}$$

With a completely dry initial condition $\Theta_i^\ddagger = 0$, equation (B.2) yields $Q = \rho$.

Expressions for u_0 and u'_0/u_0

$$u_0 = 0.5\exp\left(-\zeta^2/\tau\right)\left(a + b - c + d - e\right) \tag{B.3}$$

where:

$$a = 2\exp\left(\left(\zeta - Q\tau\right)/\sqrt{\tau}\right)^2 \tag{B.4}$$

$$b = f\left(\left(\zeta - g\tau\right)/\sqrt{\tau}\right) \tag{B.5}$$

$$c = f\left(\left(\zeta - Q\tau\right)/\sqrt{\tau}\right) \tag{B.6}$$

$$d = f\left(\left(\zeta + g\tau\right)/\sqrt{\tau}\right) \tag{B.7}$$

$$e = f\left(\left(\zeta + Q\tau\right)/\sqrt{\tau}\right) \tag{B.8}$$

$$g = \rho\sqrt{1 + 1/\rho} \tag{B.9}$$

$$f(x) = \exp(x^2)\,\mathrm{erfc}(x) \tag{B.10}$$

Differentiating (B.3) with respect to ζ yields:

$$u'_0 = exp\left(-\zeta^2/\tau\right)\left(Q(a+c-e) - g(b-d)\right) \quad \text{(B.11)}$$

Equation (4.13) requires u'_0/u_0. Because b, in (B.5), is some orders of magnitude larger than the other terms in (B.3) and (B.11), Θ-values computed directly from these equations may slightly exceed the theoretical maximum at the trailing edge of the wetting front, using 64-bit arithmetic. By taking the quotient of (B.3) and (B.11) and rearranging the algebra, b may be eliminated from the numerator, and the numerical error is minimised. The resulting expression is:

$$u'_0/u_0 = 2\left(\frac{Q(a+c-d)+gd}{a+b-c+d-e}\right) - 2g\left(1-\frac{a-c+d-e}{a+b-c+d-e}\right) \quad \text{(B.12)}$$

Equations (B.3) and (B.12) can be used to solve (4.13) and (4.14).

Expressions for summation terms of u and u'

$$u_{s1} = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{4} S_j\left(exp(c_1)f(c_2) - exp(c_3)f(c_4)\right) \quad \text{(B.13)}$$

$$u_{s2} = \sum_{n=0}^{\infty}\sum_{j=1}^{3} \omega_j\left(exp(c_5)f(c_6) - exp(c_7)f(c_8)\right) \quad \text{(B.14)}$$

where:

$$c_1 = 4n^2\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger}\rho - \left(2n\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} + \zeta\right)^2/\tau \quad \text{(B.15)}$$

$$c_2 = \left(2n\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} + \zeta\right)/\sqrt{\tau} - \left(2n\rho + b_j\right)\sqrt{\tau} \quad \text{(B.16)}$$

$$c_3 = 4n^2\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger}\rho - \left(2n\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} - \zeta\right)^2/\tau \quad \text{(B.17)}$$

$$c_4 = \left(2n\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} - \zeta\right)/\sqrt{\tau} - \left(2n\rho - b_j\right)\sqrt{\tau} \quad \text{(B.18)}$$

$$c_5 = \left(4n^2+4n\right)\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger}\rho - \left((2n+1)\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} + \zeta\right)^2/\tau \quad \text{(B.19)}$$

$$c_6 = \left((2n+1)\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} + \zeta\right)/\sqrt{\tau} - \left(2n\rho + a_j\right)\sqrt{\tau} \quad \text{(B.20)}$$

$$c_7 = \left(4n^2+4n\right)\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger}\rho - \left((2n+1)\left(1-\Theta_i^{\ddagger}\right)\gamma^{\ddagger} - \zeta\right)^2/\tau \quad \text{(B.21)}$$

$$c_8 = \left((2n+1)\left(1-\Theta_i^\ddagger\right)l^\ddagger - \zeta\right)/\sqrt{\tau} - \left(2n\rho + a_j\right)\sqrt{\tau} \quad \text{(B.22)}$$

$$\omega_j = \left[-exp\left(2\left(1-\Theta_i^\ddagger\right)l^\ddagger Q\right) - \omega_1/2, -\omega_1/2\right] \quad \text{(B.23)}$$

$$S_j = [+1, +1, -1, -1] \quad \text{(B.24)}$$

$$a_j = [\rho, 2\rho, -Q, Q] \quad \text{(B.25)}$$

$$b_j = \left[\sqrt{\rho(\rho+1)}, -\sqrt{\rho(\rho+1)}, Q, -Q\right] \quad \text{(B.26)}$$

The derivatives of the summation terms with respect to ζ are:

$$u'_{s1} = \sum_{n=1}^{\infty}\sum_{j=1}^{4} -S_j\left(2n\rho + b_j\right)\left(exp(c_1)f(c_2) + exp(c_3)f(c_4)\right) \quad \text{(B.27)}$$

$$u'_{s2} = \sum_{n=0}^{\infty}\sum_{j=1}^{3} -2\omega_j\left(2n\rho + a_j\right)\left(exp(c_5)f(c_6) + exp(c_7)f(c_8)\right) \quad \text{(B.28)}$$

For a finite-depth profile, a test of computational roundoff error at the critical lower boundary may be made by checking the deviation of $z^\ddagger(\zeta_{max})$ from depth $l^\ddagger$, where $\zeta_{max} = (1-\Theta_i^\ddagger)l^\ddagger - \rho\tau$ represents the lower boundary at any point in time. Analytic solutions are valid in principle for $\zeta_{max} > 0$. However this test is required because summation terms in the expressions for $\partial u/\partial\zeta$ and u may be too large for accurate computation, even with 64-bit arithmetic, because the greatest number that can be represented is $10^{\pm 308}$. This problem occurs if ρ is too large in relation to $l^\ddagger$ and soil-water content is large at the lower boundary. It cannot be solved simply by rearranging the algebra (*Broadbridge*, 1990). The test provides a good direct check on the accuracy of $z^\ddagger$ as computed by (4.14), and an indirect check on the accuracy of $\Theta^\ddagger$ as computed by (4.13).

For a semi-infinite profile, no fixed value of ζ_{max} will correspond to an arbitrary scaled depth $l^\ddagger$. For computation to that depth, ζ_{max} lies in the range of $(1-\Theta_i^\ddagger)l^\ddagger - \rho\tau \le \zeta_{max} \le l^\ddagger$. In general, the ζ-range must be searched to obtain a solution for a given value of $z^\ddagger$.