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The accumulation of palaeomagnetic results from multicomponent analyses

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Summary. In palaeomagnetic studies the analysis of multicomponent magnetizations has evolved from the eye-ball, orthogonal plot, and vector difference methods to the more elaborate computer-based methods such as principle component analysis (PCA), linearity spectrum analysis (LSA), and the recent package called LINEFIND. The errors involved in estimating a particular direction in a multicomponent system from a single specimen are fundamental to PCA, LSA, and LINEFIND, yet these errors are not used in estimating an overall direction from a number of observations of a particular component (other than in some acceptance or rejection criterion). The distribution of errors relates very simply to a Fisher distribution, and so these errors may be included fairly naturally in the overall analysis. In the absence of a rigorous theory to cover all situations, we consider here approximate methods for the use of these errors in estimating overall directions and cones of confidence. Some examples are presented to demonstrate the application of these methods.

Key words: palaeomagnetism, multicomponent magnetizations, data analysis

1 Introduction

In the early years of palaeomagnetic investigations it was often assumed that specimens contained, at most, one useful palaeodirection, even though Creer (1955, 1957) had shown the existence of three magnetic components in the Sidmouth Keuper Marls. Furthermore, the direction of natural remanent magnetization (NRM) was often taken as a representation of the palaeodirection. Alternating field demagnetizations were carried out by Brynjolfsson (1957), Creer (1958, 1959), and As & Zijderveld (1958), and it subsequently became recognized that the NRM direction was usually a poor representation of the desired palaeodirection. It then became common practice, whenever possible, to isolate an 'endpoint' using alternating field or thermal demagnetization, and to use this endpoint as a representation of

the palaeodirection. As demagnetization techniques became more widespread and more sophisticated, it was generally recognized that rock specimens (particularly sediments) frequently contained more than one useful palaeodirection, and the requirement for multi-component analysis became well-recognized. Initially, fairly simplistic (yet often quite effective) techniques were developed such as stereograms and J–H plots (Creer 1958), orthogonal graphs (Zijderveld 1967), vector subtraction (Roy & Parks 1974), circles of magnetization (Creer 1955; Khramov 1958; Jones, Robertson & McFadden 1975; Halls 1976, 1978; McFadden 1977) and difference vector circles (Hoffman & Day 1978). Creer (1962) introduced a statistical test for coplanarity of observations, but the first attempt to provide a genuine statistical basis for multicomponent analysis was Kirschvink (1980) who adapted principle component analysis (PCA) to palaeomagnetic data. Schmidt (1982) then extended Kirschvink's PCA and introduced a technique whereby individual specimens were not analysed in isolation, but data from samples, or specimens, of a homogeneous group were considered together. Schmidt (1982) referred to his technique as linearity spectrum analysis (LSA). Recently, Kent, Briden & Mardia (1983) have presented a realistic statistical model for the multicomponent analysis of a single specimen in isolation; the analysis is implemented by a computer program they call LINEFIND.

Clearly the most important problem is the correct identification of linear segments in the step-wise demagnetization data, and a satisfactory statistical analysis is dependent upon such correct identification. However, even though the errors involved in estimating a particular multicomponent direction are fundamental to PCA, LSA and LINEFIND, these errors are not at present used in estimating an overall direction from a number of specimens, nor in estimating a cone of confidence about this direction. In this paper an attempt is made to compare the statistical bases of LINEFIND and LSA, to relate the results of both to the Fisher distribution, and to use the error analyses in estimating precisions, overall directions, and cones of confidence. On occasion this will require combination of observations from populations with different precisions, and although a rigorous analysis exists for the large sample situation (see Fisher & Lewis 1983; Watson 1983) there is not as yet a rigorous analysis for the small sample case. However, there is a clear need for some method of estimating population precision parameters and cones of confidence. Consequently, in the absence of a rigorous theory, we provide here an approximate analysis for combining results from multicomponent analyses. Finally, we provide some examples.

2 Comparison of LINEFIND and LSA

The notation used here is similar to that used by Kent *et al.* (1983). Their analysis and the analyses of Kirschvink (1980) and Schmidt (1982) are assumed known, but adequate information on these analyses is presented here for understanding this paper.

2.1 LINEFIND

At each stage of demagnetization the remanent magnetization is measured as a vector in three-dimensional space, and so the observations can be represented as a sequence of vectors $\mathbf{x}_i$ in R^3. For a particular specimen assume that it contains a magnetization in the direction of the vector $\mathbf{T}$. Furthermore, assume that by using LINEFIND a segment of $(m-1)$ demagnetization steps has been identified during which only this particular component has been demagnetized, and represent the m remanence vectors associated with this segment by $\mathbf{x}_i$ $(i = 1, \ldots, m)$. A straight line fitted to the ends of these vectors will have a direction that is an estimate of the direction of $\mathbf{T}$. A fundamental aspect of the LINEFIND process is that

the statistical model assigns a variance σ^2 (assumed known either through a predictive model of the errors or by multiple measurements at each demagnetization step) to each of the remanence vectors $\mathbf{x}_i$. The weighted sums of squares and products matrix, **S**, is then defined by

$$\mathbf{S} = \sum_{i=1}^{m} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^t / \sigma_i^2, \tag{1}$$

where $\bar{\mathbf{x}}$ is the weighted (according to the variances) mean and $(\cdot)^t$ denotes the transpose of $(\cdot)$ [see equations (5) and (6) of Kent *et al.* 1983]. The matrix **S** will, in general, have three real positive eigenvalues. Let the largest of these eigenvalues be λ and let the unit eigenvector associated with λ be **V**. The direction of **V** is then the least squares estimate of the direction of **T** and, as shown by Kent *et al.* (1983), if ϑ is the angle between **V** and **T** then, asymptotically,

$$\lambda(1 - \cos\vartheta) \sim \chi_2^2 \tag{2}$$

The symbol '~' is to be read as meaning 'is (approximately) distributed as' and χ_ν^2 is the chi-square distribution on ν degrees of freedom.

If we now take a Fisher distribution (Fisher 1953) with precision κ and if ϑ is the angle between a random observation and the true mean direction of the population then the density, $P(\vartheta)$, of ϑ is given by

$$P(\vartheta)\, d\vartheta = \frac{\kappa}{2 \sinh \kappa} \exp(\kappa \cos\vartheta) \sin\vartheta\, d\vartheta. \tag{3}$$

For large values of κ (e.g. $\kappa > 3$), $2 \sinh \kappa \approx \exp(\kappa)$ and so

$$P(\vartheta)\, d\vartheta \approx \kappa \exp[-\kappa(1 - \cos\vartheta)] \sin\vartheta\, d\vartheta. \tag{4}$$

Letting

$$U = 2\kappa(1 - \cos\vartheta)$$

we have that

$$dU = 2\kappa \sin\vartheta\, d\vartheta$$

and substitution of this into (4) gives

$$P(U)\, dU \approx \tfrac{1}{2} \exp(-\tfrac{1}{2} U)\, dU. \tag{5}$$

Hence

$$2\kappa(1 - \cos\vartheta) = U \sim \chi_2^2. \tag{6}$$

Comparison of equations (2) and (6) shows that the direction of the eigenvector **V** may be considered as a random observation from a Fisher distribution with true mean direction given by **T** and with precision κ given by

$$\kappa = \tfrac{1}{2}\lambda. \tag{7}$$

It should be noted that from LINEFIND this precision κ is, in effect, known.

From equation (2) an approximate $100(1-p)$ per cent confidence region for the direction of **T** is given by a cone of semi-angle

$$\vartheta_{1-p} = \arccos\left(1 + \frac{2 \ln(p)}{\lambda}\right) \tag{8}$$

about the direction of **V**. This result is given by Kent *et al.* (1983) as their equation (13).

2.2 LSA

Assume now that the analysis has been performed by LSA and that again a segment of $(m-1)$ demagnetization steps has been identified during which only the component with direction **T** has been demagnetized. For both PCA and LSA the matrix analysed is the unweighted sums of squares and products matrix, **H**, defined by

$$\mathbf{H} = \sum_{i=1}^{m} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^t, \tag{9}$$

where $\bar{\mathbf{x}}$ is now the unweighted mean. Use of the unweighted sums of squares and products is effectively equivalent to assuming a constant variance in the measured remanence vectors $\mathbf{x}_i$, i.e. it is equivalent to assuming that

$\sigma_i^2 = \sigma^2$ for all i in $1, \ldots, m$

and σ^2 has to be estimated from the matrix **H**. This matrix **H** will have three real positive eigenvalues e_1, e_2 and e_3. Assuming that the vectors $\mathbf{x}_i$ are sufficiently different that the ends of these vectors define a line not buried in noise, then $e_1 > e_2 \geqslant e_3$ and the eigenvector **V** associated with the eigenvalue e_1 is an estimate of the direction of T. It is worth noting that if $\langle \mathbf{H} \rangle$ has eigenvalues ϵ_i, then $\epsilon_2 = \epsilon_3$, where $\langle . \rangle$ denotes expectation. Consider now a matrix $\mathbf{S}'$ that is similar to matrix **S**, but with a common variance σ^2 instead of the variable variance σ_i^2. The matrix $\mathbf{S}'$ will then have three real positive eigenvectors $l_1 > l_2 \geqslant l_3$, which are related to the eigenvalues of **H** by

$$l_j = e_j/\sigma^2 \,;\; j = 1, 2, 3. \tag{10}$$

Identifying l_1 of $\mathbf{S}'$ with λ of **S**, equation (7) then indicates that the eigenvector **V** may be considered as a random observation from a Fisher distribution with true mean direction **T** and with precision κ given by

$$\kappa = \frac{e_1}{2\sigma^2} \,. \tag{11}$$

Unfortunately σ^2 is not known in this instance and must be estimated from the data set through the matrix **H**.

Asymptotically,

$$(l_2 + l_3) \sim \chi^2_{2(m-1)} \cdot$$

and so

$$\frac{e_2 + e_3}{\sigma^2} \sim \chi^2_{2(m-1)} \tag{12}$$

Thus the eigenvalues e_2 and e_3 can be used to estimate σ^2. From equation (2) (with l_1 instead of λ) we also have that

$$\frac{e_1(1 - \cos\vartheta)}{\sigma^2} \sim \chi^2_2 \tag{13}$$

and asymptotically the statistics of (12) and (13) are independent. Thus the unknown σ^2 may be eliminated by noting that

$$\frac{e_1(1 - \cos\vartheta)}{2\sigma^2} \Big/ \frac{2(m-1)\sigma^2}{e_2 + e_3} \sim F[2; 2(m-1)]$$

giving us that

$$\frac{e_1}{e_2 + e_3}(m-1)(1-\cos\vartheta) \sim F[2;2(m-1)], \tag{14}$$

where $F[\nu_1;\nu_2]$ is the F-distribution on ν_1 and ν_2 degrees of freedom. Integrating out the F-distribution then gives an approximate $100(1-p)$ per cent confidence region for the direction of T as a cone of semi-angle ϑ_{1-p} about the direction of V where

$$\cos\vartheta_{1-p} = 1 - \frac{e_2 + e_3}{e_1}\left[\left(\frac{1}{p}\right)^{1/(m-1)} - 1\right]. \tag{15}$$

From (11) and (12) it follows that

$$2\kappa\,\frac{e_2 + e_3}{e_1} \sim \chi^2_{2(m-1)}$$

giving

$$\left\langle \frac{e_2 + e_3}{e_1} \right\rangle \approx \frac{m-1}{\kappa}.$$

Hence, putting

$$k = \frac{(m-1)e_1}{e_2 + e_3} \tag{16}$$

we have that

$$\left\langle \frac{1}{k} \right\rangle \approx \frac{1}{\kappa}$$

showing that k is a useful estimator for κ in this situation. Clearly $(1/k)$ bears a close relation to the MAD (maximum angular deviation) parameter used by Kirschvink (1980) and Schmidt (1982) but is a more appropriate statistic.

2.3 COMPARISON

From a statistical point of view LINEFIND is substantially more sophisticated than LSA. Implementation of LINEFIND requires more data as input to allow determination of the variances σ_i^2 but, quite naturally, more information is available as a result. Not surprisingly, LINEFIND is more expensive in computing time than LSA. As a result of the extra information available at the specimen level it is possible to test, without reference to any external information, whether a straight line adequately describes a segment of remanence vectors $\mathbf{x}_i$. Physically, this is equivalent to stating that it is possible to test at the specimen level whether the hypothesis 'that only a single direction of magnetization is being removed within a specified segment of the demagnetization sequence' adequately describes the data obtained. However, because the analysis is performed entirely at the level of an individual specimen, without reference to any other specimens, it is possible to obtain a statistically significant linear segment which is not obtained from any other specimen analysed. The geological interpretation of such a component is then difficult.

In contrast, with LSA it is not possible to test whether a straight line provides a statistically adequate description of a given segment of remanence vectors $\mathbf{x}_i$. Instead an

arbitrary decision is made on an 'acceptable linearity' by setting a critical value for the MAD parameter (or equivalently a critical value for k, the estimate of κ, for a given value of m). If a particular segment of remanence vectors $\mathbf{x}_i$ passes this 'linearity test' analysis then proceeds on the assumptions that (a) a straight line does provide an adequate description and (b) the observations $\mathbf{x}_i$ share a common variance σ^2. The analysis of Section 2.2 is also predicated on these two assumptions. Proceeding on the basis of these two assumptions is somewhat akin to the most common use of simple linear regression (in which it is typically assumed on the basis of some model that a straight line does provide an adequate description of the data and that the observations do share a common variance) – provided the 'signal/noise' ratio is large adequate results are frequently achieved at minimal cost. The major problem arises when the 'signal/noise' ratio is not particularly large because it is not possible to test whether the results are in fact adequate – faith in such adequacy has to be achieved by making reference to some additional, external information. With LSA such faith in the adequacy of results relies on a common palaeomagnetic sampling scheme in which several specimens are derived from a homogeneous source. A commonality of information from several specimens then enhances one's faith in the geological reality of identified components. However, there will always be more information available at the specimen level with LINEFIND than with LSA, and this relative lack of information in the LSA results is indicated in the extra complexity of the analysis in Section 2.2 compared with Section 2.1.

Clearly the LINEFIND algorithm and that part of the LSA algorithm which compares the linearity spectra of the different specimens (thereby enhancing confidence in the geological reality of identified components) are not incompatible. Thus it may be that, when analysing several specimens from a homogeneous source, an optimal algorithm would be a dovetailing of LINEFIND with the interspecimen comparison of LSA.

2.4 RELIABILITY OF THE INDIVIDUAL PRECISION ESTIMATES

In order to simplify the analysis it will be assumed throughout Section 3 that λ or e_1 is known for each specimen and each component (as was done by Kent *et al.* 1983). On this basis, if LINEFIND was used, the precision κ for each specimen direction should be known from equation (7) as

$$\kappa = \tfrac{1}{2}\lambda.$$

However, this is true only if LINEFIND has correctly identified the endpoints of the linear segment. If endpoints have been incorrectly identified (e.g. external to the actual linear segment) then additional error will have been introduced and the direction of **T** will in fact have been identified with a precision of only κ_d where $\kappa_d^{-1} = \kappa^{-1} + \kappa_e^{-1}$, the parameter κ_e being associated with the 'endpoint error'. In order to test this, the LINEFIND tuning parameter may be varied over a fairly wide range, and if the resulting scatter of directions for a particular component is consistent with the precision κ then the analysis may proceed as outlined in Section 3. However, if the scatter is too large to be associated with the precision κ, then it must be recognized that 'endpoint error' has been introduced. Under these circumstances it cannot be considered that the precision is known, the analysis in Section 3 is not valid, and the results from different specimens should simply be combined with equal weight. Unfortunately one is not then in a position to ascribe and estimate the different sources of error.

With LSA the parameter κ is not known but must be estimated by k, given by equation (16) as

$$k = (m-1)e_1/(e_2 + e_3),$$

because the variance, σ^2, is not known and has to be estimated. From equation (12) it is seen that σ^2 is estimated through a chi-square distribution with $2(m-1)$ degrees of freedom. If the number of degrees of freedom is large then it will be reasonable to set $\kappa = k$ and assume κ to be known. However, with a small number of degrees of freedom this is not so and it cannot be assumed that κ is known. As above, the results from different specimens should then simply be combined with equal weight. It should also be recognized that the above comments about 'endpoint error' apply equally to LSA.

3 Accumulation of individual specimen results

3.1 ASSUMED MODEL

Throughout this section attention is centred on a single component that has been identified in N different specimens, and it should be noted that the index i is now used to refer to these different specimens. At this stage then the assumed model for the particular component is that there are N directions given by the vectors $\mathbf{T}_i$ and that these N directions are random observations from a Fisher distribution with mean direction given by $\boldsymbol{\mu}$ and precision κ_p. Each of the $\mathbf{T}_i$ directions is itself estimated by the direction of a unit vector $\mathbf{V}_i$ (resulting from either LINEFIND or LSA) which, for a given i, is a random observation from a Fisher distribution with mean direction given by $\mathbf{T}_i$ and precision κ_i. Furthermore, it is assumed that the κ_i are known (see Section 2). The aim of the following (approximate) analysis is to estimate the direction $\boldsymbol{\mu}$, determine an approximate cone of confidence about this estimated direction and, where possible, estimate κ_p.

3.2 DEFINITIONS AND TESTS

Let x_i, y_i, and z_i denote the coordinates of the unit vectors $\mathbf{V}_i$ so that

$\mathbf{V}_i = (x_i, y_i, z_i)^t$.

For ease of notation let Σ denote the sum for $i = 1, \ldots, N$ and define

$\tau_i^2 = 1/\kappa_i$,

$\tau_p^2 = 1/\kappa_p$,

$R = |\Sigma \mathbf{V}_i| = \sqrt{(\Sigma x_i)^2 + (\Sigma y_i)^2 + (\Sigma z_i)^2}$,

and

$R_w = |\Sigma \kappa_i \mathbf{V}_i| = \sqrt{(\Sigma \kappa_i x_i)^2 + (\Sigma \kappa_i y_i)^2 + (\Sigma \kappa_i z_i)^2}$.

Using the notation $P(\mathbf{V}) = \mathbf{V}/|\mathbf{V}|$, define the following three possible mean unit vectors

$$\bar{V}_0 = P(\Sigma \mathbf{V}_i) = \frac{1}{R} \Sigma \mathbf{V}_i \tag{17}$$

$$\bar{V}_A = P(\Sigma \kappa_i \mathbf{V}_i) = \frac{1}{R_w} \Sigma \kappa_i \mathbf{V}_i \tag{18}$$

$$\bar{V}_B = P(\Sigma \mathbf{V}_i / (\tau_i^2 + \tau_p^2)). \tag{19}$$

$\bar{V}_0$ is then the simple unweighted mean direction, $\bar{V}_A$ is the optimal estimate of $\boldsymbol{\mu}$ assuming $\tau_p^2 = 0$, and $\bar{V}_B$ is the optimal estimate assuming $\tau_p^2 \neq 0$. Naturally $\bar{V}_B$ and $\bar{V}_A$ reduce to $\bar{V}_0$ if $\kappa_i = \kappa_c$ for all i.

Also, let

$$S_0^2 = 2\Sigma\,(1 - \mathbf{V}_i^{\mathrm{t}}\bar{V}_0) = 2(N - R) \tag{20}$$

and

$$S_{\mathrm{A}}^2 = 2\Sigma\kappa_i\,(1 - \mathbf{V}_i^{\mathrm{t}}\bar{V}_{\mathrm{A}}) = 2\Sigma\kappa_i - 2R_{\mathrm{w}}. \tag{21}$$

Again, if $\kappa_i = \kappa_{\mathrm{c}}$ for all i, we have that $S_{\mathrm{A}}^2 = 2\kappa_{\mathrm{c}}(N - R)$.

3.2.1 κ_{p} large compared with the κ_i

In this situation most of the dispersion is caused by errors in estimating the directions $\mathbf{T}_i$. Although this situation is easily handled it is hoped that it would be rare since, typically, it will indicate poor experimental technique. Effectively, from a statistical point of view, the $\mathbf{T}_i$ are not themselves dispersed so $\kappa_{\mathrm{p}} = \infty$ ($\tau_{\mathrm{p}}^2 = 0$) and cannot be estimated.

Under the null hypothesis

$$H_0 : \tau_{\mathrm{p}}^2 = 0$$

we have that

$$S_{\mathrm{A}}^2 = 2\Sigma\kappa_i - 2R_{\mathrm{w}} \sim \chi^2_{2(N-1)}, \tag{22}$$

If the value of this statistic exceeds the critical value (typically the 95 per cent value) of the chi-square distribution then the null hypothesis may be rejected. Otherwise the null hypothesis may be accepted and estimation of μ performed as follows on the basis that $\tau_{\mathrm{p}}^2 = 0$.

Given that $\tau_{\mathrm{p}}^2 = 0$, the optimal estimate, $\hat{\mu}$, for μ is given by $\bar{V}_{\mathrm{A}}$, i.e.

$$\hat{\mu} = \bar{V}_{\mathrm{A}} = \frac{1}{R_{\mathrm{w}}}\,\Sigma\kappa_i\,\mathbf{V}_i. \tag{23}$$

Furthermore, since $\tau_{\mathrm{p}}^2 = 0$, the individual dispersions are known, so $\bar{V}_{\mathrm{A}}$ estimates μ with known precision κ_{A} given by

$$\kappa_{\mathrm{A}} = \Sigma\,\kappa_i. \tag{24}$$

If α is the angle between the estimate $\bar{V}_{\mathrm{A}}$ and μ, i.e.

$$\cos\alpha = \bar{V}_{\mathrm{A}} \cdot \mu, \tag{25}$$

then it follows (refer to equation 6) that

$$2\kappa_{\mathrm{A}}\,(1 - \cos\alpha) \sim \chi_2^2. \tag{26}$$

Thus an approximate $100(1 - p)$ per cent confidence region for the direction of μ is given by a cone of semi-angle α_{1-p} about the direction of $\bar{V}_{\mathrm{A}}$ where

$$\cos\alpha_{1-p} = 1 + \frac{\ln(p)}{\kappa_{\mathrm{A}}}. \tag{27}$$

Typically $p = 0.05$ for a 95 per cent confidence region giving

$$\cos\alpha_{0.95} = 1 + \frac{\ln(0.05)}{\kappa_{\mathrm{A}}}.$$

3.2.2 κ_p not large compared with the κ_i

If H_0 has been rejected then $\tau_p^2 \neq 0$ and the optimal estimate of μ is $\bar{V}_B$. However, τ_p^2 is unknown and so $\bar{V}_B$ cannot be obtained.

Each of the $\mathbf{V}_i$ will now be (approximately) distributed about μ with a precision κ_i' where

$$\frac{1}{\kappa_i'} = \tau_p^2 + \tau_i^2 .$$

In this situation, addition of τ_p^2 to the τ_i^2 will tend to smooth out differences between the τ_i^2 and the κ_i' will tend to be roughly equal. Consequently it will be sufficient to suppose that all of the observations were drawn from a single Fisher distribution with precision κ^* where

$$\frac{1}{\kappa^*} = \tau_p^2 + \frac{1}{N}\Sigma\tau_i^2 . \tag{28}$$

On this basis we would set the estimate, $\hat{\mu}$, of μ equal to $\bar{V}_0$, and assess the accuracy of $\bar{V}_0$ using the usual Fisher statistics. The precision κ^* would then be estimated in the usual manner by k^* where

$$k^* = \frac{N-1}{N-R} \tag{29}$$

and combination of this with equation (28) provides a natural estimate $k_p (= 1/t_p^2)$ of κ_p with

$$\frac{1}{k_p} = t_p^2 = \frac{N-R}{N-1} - \frac{1}{N}\Sigma\tau_i^2 = \frac{S_0^2}{2(N-1)} - \frac{1}{N}\Sigma\tau_i^2 . \tag{30}$$

If $\kappa_i = \kappa_c$ (i.e. $\tau_i^2 = \tau_c^2$) for all i then, as shown by Watson (1956),

$$\frac{S_0^2}{\tau_p^2 + \tau_c^2} \sim \chi^2_{2(N-1)}, \tag{31}$$

so that

$$2(N-1)\frac{t_p^2 + \tau_c^2}{\tau_p^2 + \tau_c^2} \sim \chi^2_{2(N-1)}, \tag{32}$$

from which confidence limits for τ_p^2 or κ_p may be obtained. If the τ_i^2 are unequal then

$$2(N-1)\frac{t_p^2 + \frac{1}{N}\Sigma\tau_i^2}{\tau_p^2 + \frac{1}{N}\Sigma\tau_i^2} \cong U : U \sim \chi^2_{2(N-1)}, \tag{33}$$

the approximation becoming worse as the variability amongst the τ_i^2 increases.

4 Examples of application

In the following examples LSA has been used to define the components of magnetization present. Small variations would probably have resulted from applying LINEFIND to the data but these are considered of no consequence for the purpose of demonstration.

4.2 EXAMPLES OF $\tau_p^2 = 0$

These examples come from an analysis of the step-wise thermal demagnetization data of three specimens from the same drill core sample (JL13D) of Mt Jope volcanics. The palaeomagnetism of these rocks is discussed in detail by Schmidt & Embleton (1985). Briefly, the magnetizations consist of three components:

(i) a low temperature (200°C) component close to the present geomagnetic field and interpreted as being largely of viscous origin;

(ii) an intermediate temperature (400°C) component which is syn-deformational in age and therefore a secondary magnetization; and

(iii) a high temperature (540°C) component, pre-folding in age and therefore probably a primary magnetization.

All components identified by LSA comprise at least three data points, one of which is the point measured after heating to the temperature specified.

These three groups are listed in Table 1 where the directions of each component and their precisions, κ_i, derived from each specimen are given. The directions and their individual 95 per cent confidence areas, calculated from equation (6), are plotted in Fig. 1. For each group there is a large degree of overlap of the confidence areas and we may enquire if the κ_i dominate the dispersion, which is equivalent to $\kappa_p = \infty$ or $\tau_p^2 = 0$. From the values of the test statistic, S_A^2 listed in Table 1, this is seen to be so for each group, and so weighted mean directions, $\bar{V}_A$ (equation 18), have been calculated together with their 95 per cent confidence semi-angles (α_{95}) from equation (27). In Table 1 these results are compared with what would have been obtained by using (incorrectly) the usual unweighted Fisher statistics.

Table 1. Magnetic components of sample JL13D.

Specimen	Temp (°C)	Dec (°)	Inc (°)	κ_i
1	200	347.2	−58.2	87
2	200	349.4	−59.9	129
3	200	351.4	−57.4	556
$S_A^2 = 0.3337 < \chi_4^2\,(0.95) = 9.488$		Accept $H_0 : \tau_p^2 = 0$	Estimate μ by $\bar{V}_A$	
Correct mean, $\bar{V}_A$		350.6	−57.9	$\alpha_{95} = 5.0°$
(Unweighted mean		349.3	−58.5	$\alpha_{95} = 2.6°$)
1	400	305.7	−23.6	756
2	400	308.7	−27.7	611
3	400	308.8	−21.3	384
$S_A^2 = 4.291 < \chi_4^2(0.95) = 9.488$		Accept $H_0 : \tau_p^2 = 0$	Estimate μ by $\bar{V}_A$	
Correct mean, $\bar{V}_A$		307.4	−24.5	$\alpha_{95} = 3.4°$
(Unweighted mean		307.7	−24.2	$\alpha_{95} = 5.5°$)
1	540	48.2	67.5	4725
2	540	48.4	67.2	9219
3	540	49.0	70.9	342
$S_A^2 = 1.410 < \chi_4^2\,(0.95) = 9.488$		Accept $H_0 : \tau_p^2 = 0$	Estimate μ by $\bar{V}_A$	
Correct mean, $\bar{V}_A$		48.3	67.4	$\alpha_{95} = 1.2°$
(Unweighted mean		48.5	68.5	$\alpha_{95} = 3.1°$)

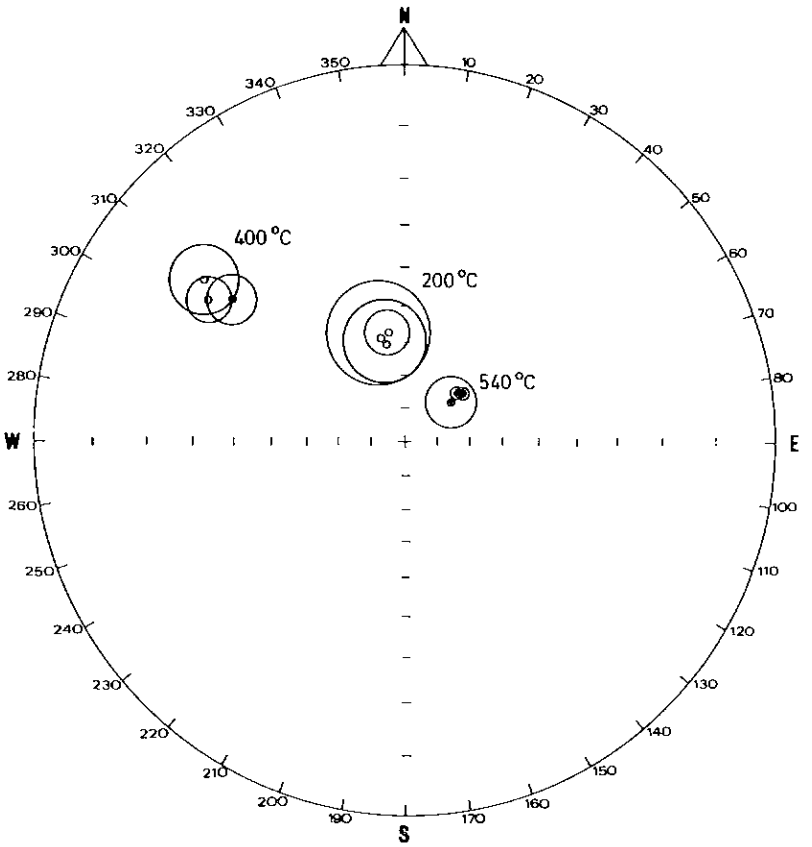

Figure 1. Directions and individual 95 per cent confidence areas for the three groups from sample JL13D. Each is an example of $\tau_p^2 = 0$.

Although the mean directions are little different, the α_{95} values do differ quite substantially. Consequently, by using this analysis, the derived values of α_{95} are more accurate, and there is a recognition that no information has been gathered about κ_p apart from the fact that it is large compared with the κ_i.

It is in fact quite unusual for the κ_i to dominate, and so it is worth commenting on the reason for it here. The examples were all derived from the same drill core of a volcanic rock, and so a very high precision κ_p is to be expected since scatter as a result of secular variation and orientation error have effectively been eliminated. Furthermore, the magnetizations are weak (being about 0.1 mA m^{-1} at 540°C and only about 1 mA m^{-1} even at NRM), tending to produce lower values of the κ_i than in other circumstances. Typically, when magnetizations are stronger (greater than 10 mA m^{-1}), the κ_i are larger and so do not contribute so much to the scatter.

4.2 EXAMPLES OF $\tau_p^2 \neq 0$

As above, these examples come from an analysis of step-wise demagnetization data of specimens from a single sample. In this instance the sample is a block sample (PQ10H) of Prospect Dolerite. The palaeomagnetism of the Prospect Dolerite has previously been

Table 2. Magnetic components of sample PQ10H.

Specimen	Treatment (mT)	Dec (°)	Inc (°)	κ_i
1	15	56.7	−68.2	>999 999
2	15	53.1	−62.4	161
3	15	44.9	−63.7	431
4	15	25.3	−68.9	872
5	15	21.7	−66.2	670
6	15	19.1	−69.9	1047
$S_A^2 = 135.0 > \chi^2_{10}(0.95) = 18.31$		Reject $H_0 : \tau_p^2 = 0$	Estimate μ by $\bar{V}_0$	
Correct mean, $\bar{V}_0$		37.7	−67.3	$\alpha_{95} = 5.9°$
$k_p = 172$ $45 < \kappa_p < 555$ (95 per cent confidence)				
	(°C)			
1	525	237.5	−77.7	2086
2	525	245.8	−79.9	6567
3	525	283.6	−72.9	122 238
4	525	277.4	−78.0	383 993
5	525	275.9	−72.0	18 799
6	525	287.9	−74.7	7081
$S_A^2 = 1110.5 > \chi^2_{10}(0.95) = 18.31$		Reject $H_0 : \tau_p^2 = 0$	Estimate μ by $\bar{V}_0$	
Correct mean, $\bar{V}_0$		270.9	−76.5	$\alpha_{95} = 4.7°$
$k_p = 207$ $66 < \kappa_p < 436$ (95 per cent confidence)				

reported by Boesen, Irving & Robertson (1961) and Schmidt (1982). The latter study showed the two component nature of the remanence, and in fact provided the impetus for developing LSA. The treatment of the specimens examined here consisted of various alternating field demagnetization steps up to 15 mT, followed by step-wise thermal demagnetization from 200°C up to 550°C.

LSA of the data yielded the components listed in Table 2 and plotted in Fig. 2. In each case the test statistic S_A^2 clearly exceeds the critical value of the chi-square distribution and so the hypothesis that $\tau_p^2 = 0$ can be rejected. In both cases therefore the usual unweighted Fisher statistics can be used to determine the mean direction and cone of confidence about the mean. However, an estimate k_p of κ_p can now be obtained from the data using equation (30) and confidence limits for κ_p obtained from equation (33). These values are also presented in Table 2.

4.2.1 Cases in which κ_p dominates

For the 525°C component from sample PQ10H (Table 2 and Fig. 2) the estimate κ_p is 207, much less than any of the κ_i, and so it may be considered that the individual directions resulting from the multicomponent analysis are precisely defined, the dispersion being dominated by κ_p. In such circumstances k^* (the standard estimate of a Fisher precision parameter, see equation 29) is effectively equal to κ_p. In this particular example, $k^* = 201$ compared with $k_p = 207$.

Domination by k_p comes about because the dispersion is due almost entirely to effects other than measurement error or demagnetization technique. Consequently it is to be hoped, and expected, that most analyses of palaeomagnetic data involving LSA or LINEFIND will

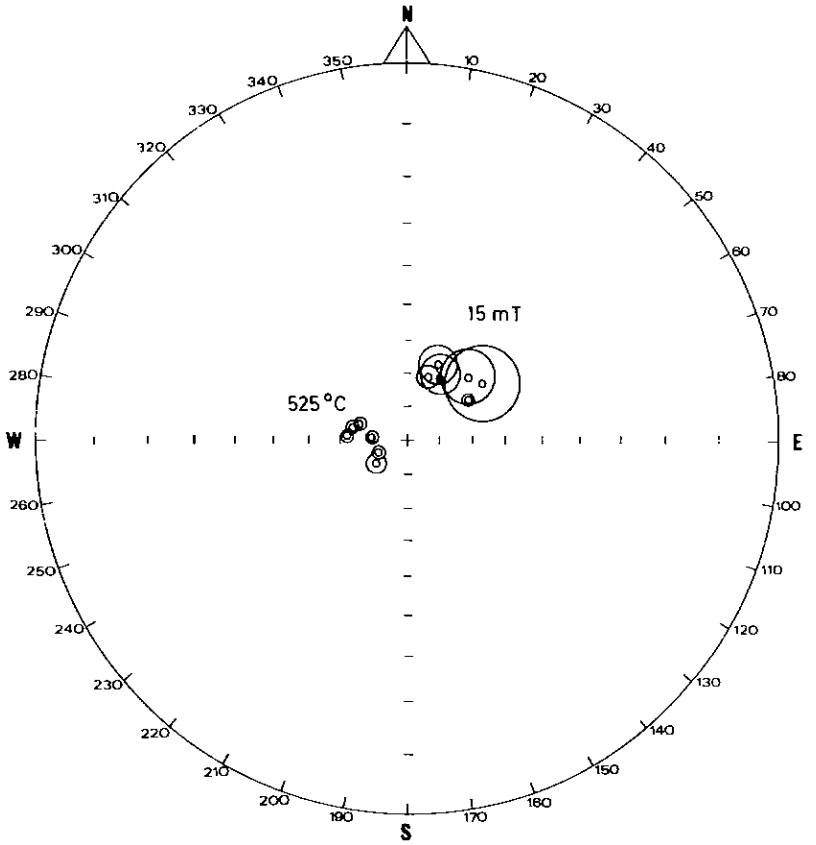

Figure 2. Directions and individual 95 per cent confidence areas for the two groups from sample PQ10H. Both groups are examples of $\tau_p^2 \neq 0$ and the 525°C group is an example of the dispersion from κ_p dominating.

fall into this class. Certainly in our search for examples this was by far the most common situation.

5 Discussion

Quite commonly the directions resulting from multicomponent analyses can be considered as being precisely defined, compared with the dispersion to be expected from sources such as secular variation and orientation errors. However, by applying the simple analysis suggested here, it is possible to test whether this is in fact so, rather than merely having to assume it to be the case. In instances where it is not the case, the analysis may be used to obtain cones of confidence, estimates of true directions, and estimates of the dispersion from the different sources. However, it must be recognized that the analysis is only approximate, the degree of approximation being substantially more if applied to LSA results than if applied to LINEFIND results.

One point of interest, which may be seen in the examples given, is that it is not uncommon to have values of κ_i (the precision with which a particular specimen direction is estimated by a multicomponent analysis) in excess of 1000. Compared with other sources of dispersion in palaeomagnetism, this dispersion is insignificant, and so quite often it will

matter little what method of analysis is used to define the direction of a particular component of magnetization. By far the most important aspect is the correct identification of linear segments in the step-wise demagnetization data. The analysis presented does not take into account errors caused by misidentification of linear segments.

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