Magnetic Tensor Gradiometry in the Marine Environment: Correction of Electric And Magnetic Field And Gradient Measurements in a Conductive Medium and Improved Methods for Magnetic Target Location Using the Magnetic Gradient Tensor.

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Introduction

The CSIRO Division of Materials Science and Engineering is developing sensitive magnetic tensor gradiometers, based on high-T SQUID technology, for deployment in the marine environment. The applications include gradient measurements as an adjunct to E and B field measurements in marine CSEM surveys, UXO detection in shallow water, and exploration for seafloor mineralization. This paper discusses how measurements of quasistatic electric and magnetic fields and their gradients in the ocean are affected by electric current flow in the conductive medium, which is distorted by insulating capsules that enclose sensors and their associated electronics. We also present simple new methods for direct inversion of gradient tensor data for the location and magnetic moment vector of compact targets

Electric and magnetic fields in and around an insulating spherical capsule

Electric and magnetic fields within a medium of conductivity σ are perturbed by the measurement process. In particular, sensors located within or around an insulating measurement capsule measure fields that are modified by the diversion of conduction currents around the capsule. In air or free space the gradient tensor is symmetric, as well as traceless. In the presence of conduction currents the curl of **B** is non-zero and the gradient tensor is asymmetric. This raises the question of what is actually measured by magnetometers and gradiometers immersed in the electrically conductive ocean. In particular, how does the signal measured within a sealed capsule (within which the gradient tensor is symmetric) relate to the field components and the asymmetric gradient tensor that existed in the surrounding medium prior to insertion of the measurement package?

If a uniform applied electric field gradient is present, the unperturbed field is given by:

$$\mathbf{E}_0(\mathbf{r}) = \overline{\mathbf{E}}_0 + \nabla\mathbf{E}_0.\mathbf{r}, \tag{1}$$

where $\overline{\mathbf{E}}_0 = \overline{E}_0\hat{\mathbf{x}}$ is the average electric field over a volume symmetrically disposed about the origin and the electric gradient tensor $\nabla\mathbf{E}_0 = [\partial E_j/\partial x_i] = [E_{ij}]$,($i, j = x, y, z$) is symmetric and traceless. The corresponding unperturbed potential is:

$$V_0(\mathbf{r}) = -\overline{\mathbf{E}}_0.\mathbf{r} - \frac{1}{2}\mathbf{r}.\nabla\mathbf{E}_0.\mathbf{r}. \tag{2}$$

If a spherical cavity of radius *a* is inserted into the unperturbed current flow, the solution of the Neumann boundary value problem for the potential is (Clark, 2009a):

$$V(\mathbf{r}) = -\frac{3}{2}\overline{\mathbf{E}}_0.\mathbf{r} - \frac{5}{6}\mathbf{r}.\nabla\mathbf{E}_0.\mathbf{r}, \qquad (r \le a) \tag{3}$$

$$V(\mathbf{r}) = -\overline{\mathbf{E}}_0.\mathbf{r}\left(1+\frac{a^3}{2r^3}\right) - \frac{1}{2}\mathbf{r}.\nabla\mathbf{E}_0.\mathbf{r}\left(1+\frac{2a^5}{3r^5}\right). \qquad (r \ge a) \tag{4}$$

The corresponding internal field is:

$$\mathbf{E}(\mathbf{r}) = \frac{3}{2}\overline{\mathbf{E}}_0 + \frac{5}{3}\nabla\mathbf{E}_0.\mathbf{r}, \tag{5}$$

which shows that the average electric field within the cavity is equal to 1.5 times the unperturbed field that existed at the location of the cavity centre, prior to its emplacement, and that the electric field gradient is constant within the cavity and is amplified by 5/3 compared to the applied gradient. The anomalous external field due to the cavity has a quadrupole term associated with the applied gradient, which supplements the dipole field that is associated with the average applied field over the cavity. The corresponding results for the anomalous magnetic field components are:

$$\left.\begin{aligned}
B'_x &= \frac{\mu_0\sigma}{3}\left[\left(E_{zz}-E_{yy}\right)yz - E_{xy}xz + E_{xz}xy + E_{yz}(y^2-z^2)\right] \\
B'_y &= \mu_0\sigma\left[\frac{\overline{E}_0 z}{2} + \frac{\left(E_{xx}-E_{zz}\right)xz + E_{xy}yz - E_{xz}(x^2-z^2) - E_{yz}xy}{3}\right] \\
B'_z &= \mu_0\sigma\left[-\frac{\overline{E}_0 y}{2} + \frac{\left(E_{yy}-E_{xx}\right)xy + E_{xy}(x^2-y^2) - E_{xz}yz + E_{yz}xz}{3}\right]
\end{aligned}\right\} \quad (r \le a), \tag{6}$$

$$\left.\begin{aligned}
B'_x &= \frac{\mu_0\sigma a^5}{3r^5}\left[\left(E_{zz}-E_{yy}\right)yz - E_{xy}xz + E_{xz}xy + E_{yz}(y^2-z^2)\right] \\
B'_y &= \frac{\mu_0\sigma a^3}{r^3}\left[\frac{\overline{E}_0 z}{2} + \frac{a^2}{3r^2}\left[\left(E_{xx}-E_{zz}\right)xz + E_{xy}yz - E_{xz}(x^2-z^2) - E_{yz}xy\right]\right] \\
B'_z &= \frac{\mu_0\sigma a^3}{r^3}\left[-\frac{\overline{E}_0 y}{2} + \frac{a^2}{3r^2}\left[\left(E_{yy}-E_{xx}\right)xy + E_{xy}(x^2-y^2) - E_{xz}yz + E_{yz}xz\right]\right]
\end{aligned}\right\} \quad (r \ge a). \tag{7}$$

At the centre of the cavity the anomalous magnetic field $\mathbf{B}' = \mathbf{0}$, so the magnetic field at this point is equal to the unperturbed magnetic field that existed at the same point in the conductive medium, prior to insertion of the measurement capsule. Within the cavity the resultant magnetic gradient tensor (the sum of the ambient unperturbed gradient tensor and the asymmetric anomalous tensor obtained by differentiating (6)) is symmetric. Second order gradients are constant within the cavity. For the case of a uniform applied field, the anomalous external electric and magnetic fields are those of an elementary current dipole, with moment $\mathbf{p} = I\Delta\mathbf{x} = -2\pi\mathbf{j}_0 a^3$, immersed in an infinite homogeneous conductive medium, in the limit as frequency goes to zero (Kraichman, 1970, p.3-2).

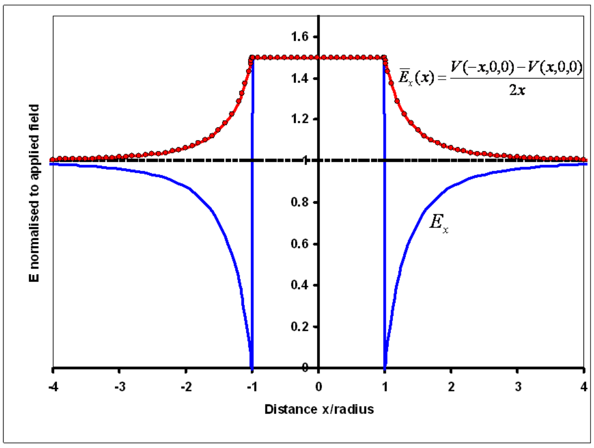

Fig.1. Electric field profile, parallel to the uniform applied field, passing through the centre of a spherical cavity.

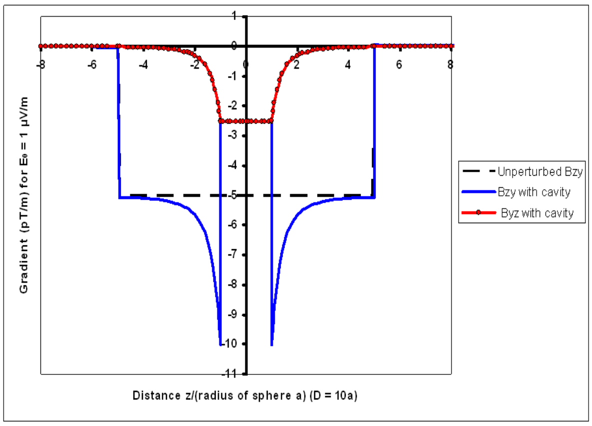

Fig.2. Magnetic gradient tensor elements along a vertical profile through the centre of a spherical cavity within a horizontal current flow distribution of limited depth extent.

Figure 1 shows the electric field along a profile parallel to a uniform applied field, passing through the centre of the cavity. **E** is discontinuous at the cavity boundary. The average value of the electric field $\overline{E}_x$ (given by the potential difference between electrodes at $\pm x$, divided by the baseline) is also shown. $\overline{E}_x$ is independent of the electric field gradient and represents the quantity measured by a standard marine electrometer, i.e.

$$\overline{E}_x(x) = \frac{V(-x,0,0) - V(x,0,0)}{2x} = \begin{cases} \overline{E}_0\left(1 + \dfrac{a^3}{2|x|^3}\right), & (|x| \geq a), \\ 1.5\overline{E}_0, & (|x| \leq a). \end{cases} \tag{8}$$

Figure 2 shows the variation of the magnetic gradient tensor elements along a vertical profile through a spherical cavity, within a horizontal current flow confined to an infinite horizontal slab of thickness equal to five times the diameter of the cavity. The assumed conductivity of the seawater is 4 Sm^{-1} and magnetic gradients are normalised to an applied electric field of 1μV/m. Within the current flow, but beyond the influence of the cavity, the only nonzero gradient tensor component is B_{zy}. Within the cavity $B_{zy} = B_{yz}$. The influence of the cavity is significant out to $r \approx 3a$.

In many marine applications electromagnetic sensors are situated near interfaces between media of differing conductivity, such as the seafloor or the sea surface. This situation can be handled by the method of images, as discussed by Clark (2009a).

Effect of an ellipsoidal cavity

Ellipsoidal cavities can be used to model a wide variety of capsule shapes, whilst conveniently allowing analytic solutions. Consider a triaxial ellipsoidal cavity, centred at the origin, with semiaxes $a > b > c$ along x_1, x_2, x_3 respectively. In terms of ellipsoidal co-ordinates ξ, η, ζ (Kellog, 1953, p.183-184; Stratton, 1941, p.58-59) the potential V_0 associated with a uniform electric field $\mathbf{E}_0$ is:

$$V_0 = -\mathbf{E}_0.\mathbf{r} = -(\mathbf{E}_0)_{x_1}\sqrt{\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(a^2-b^2)(a^2-c^2)}} - (\mathbf{E}_0)_{x_2}\sqrt{\frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{(c^2-b^2)(a^2-c^2)}} - (\mathbf{E}_0)_{x_3}\sqrt{\frac{(\xi+c^2)(\eta+c^2)(\zeta+c^2)}{(b^2-c^2)(a^2-c^2)}}. \tag{9}$$

Solving the Neumann boundary value problem in ellipsoidal co-ordinates gives for the internal potential:

$$V(\xi \le 0) = -\frac{(\mathbf{E}_0)_{x_1}}{1-D_1}x_1 - \frac{(\mathbf{E}_0)_{x_2}}{1-D_2}x_2 - \frac{(\mathbf{E}_0)_{x_3}}{1-D_3}x_3, \tag{10}$$

which implies a uniform internal field given by:

$$\mathbf{E}(\xi \le 0) = -\nabla V = \left(\frac{1}{1-D_1}\right)(\mathbf{E}_0)_{x_1}\hat{\mathbf{x}}_1 + \left(\frac{1}{1-D_2}\right)(\mathbf{E}_0)_{x_2}\hat{\mathbf{x}}_2 + \left(\frac{1}{1-D_3}\right)(\mathbf{E}_0)_{x_3}\hat{\mathbf{x}}_3. \tag{11}$$

where D_i (i = 1,2,3) are the demagnetising factors of the ellipsoid along its major, intermediate and minor axes (Clark et al., 1986). The demagnetising factors sum to unity. Since $D_1 \le D_2 \le D_3$, an ellipsoidal cavity has an anisotropic response, except in the degenerate case where all axes are equal and the cavity is spherical. Unless the applied field lies along a principal axis of the ellipsoid, the internal field is not parallel to the applied field, but is deflected away from the major axis and towards the minor axis. For a disc-like cavity $D_3 \to 1$ as $c/a \to 0$, so the amplification of the applied electric field normal to the disc can be very large within the cavity.

Define $R_s = \sqrt{(s+a^2)(s+b^2)(s+c^2)}$. The external potential due to an arbitrarily oriented ellipsoidal cavity is given by $V(\xi \ge 0) = V_1(\xi \ge 0) + V_2(\xi \ge 0) + V_3(\xi \ge 0)$, where

$$V_1(\xi \ge 0)_1 = -(\mathbf{E}_0)_{x_1} x_1\left[1 + \frac{abc/2}{1-D_1}\int_\xi^\infty \frac{ds}{(s+a^2)R_s}\right] = -(\mathbf{E}_0)_{x_1} x_1\left[1+\frac{abc/2}{1-D_1}A(\xi)\right], \tag{12}$$

$$V_2(\xi \geq 0)_1 = -(\mathbf{E}_0)_{x_2} x_2 \left[1 + \frac{abc/2}{1-D_2} \int_\xi^\infty \frac{ds}{(s+b^2)R_s} \right] = -(\mathbf{E}_0)_{x_2} x_2 \left[1 + \frac{abc/2}{1-D_2} B(\xi) \right], \qquad (13)$$

$$V_3(\xi \geq 0)_1 = -(\mathbf{E}_0)_{x_3} x_3 \left[1 + \frac{abc/2}{1-D_3} \int_\xi^\infty \frac{ds}{(s+c^2)R_s} \right] = -(\mathbf{E}_0)_{x_3} x_3 \left[1 + \frac{abc/2}{1-D_3} C(\xi) \right], \qquad (14)$$

The corresponding external field components are obtained from (12)-(14) by differentiation:

$$\begin{aligned} \mathbf{E}_{x_1} &= -\frac{\partial V(\xi \geq 0)}{\partial x_1} = (\mathbf{E}_0)_{x_1} \left\{ \left[1 + \frac{abc/2}{1-D_1} A(\xi) \right] + \frac{abcA'(\xi)x_1}{2(1-D_1)} \frac{\partial \xi}{\partial x_1} \right\} \\ &+ \left[(\mathbf{E}_0)_{x_2} \frac{abcB'(\xi)x_2}{2(1-D_2)} + (\mathbf{E}_0)_{x_3} \frac{abcC'(\xi)x_3}{2(1-D_3)} \right] \frac{\partial \xi}{\partial x_1}, \end{aligned} \qquad (15)$$

$$\begin{aligned} \mathbf{E}_{x_2} &= -\frac{\partial V(\xi \geq 0)}{\partial x_2} = (\mathbf{E}_0)_{x_2} \left\{ \left[1 + \frac{abc/2}{1-D_2} B(\xi) \right] + \frac{abcB'(\xi)x_2}{2(1-D_2)} \frac{\partial \xi}{\partial x_2} \right\} \\ &+ \left[(\mathbf{E}_0)_{x_1} \frac{abcA'(\xi)x_1}{2(1-D_1)} + (\mathbf{E}_0)_{x_3} \frac{abcC'(\xi)x_3}{2(1-D_3)} \right] \frac{\partial \xi}{\partial x_2}, \end{aligned} \qquad (16)$$

$$\begin{aligned} \mathbf{E}_{x_3} &= -\frac{\partial V(\xi \geq 0)}{\partial x_3} = (\mathbf{E}_0)_{x_3} \left\{ \left[1 + \frac{abc/2}{1-D_3} C(\xi) \right] + \frac{abcC'(\xi)x_3}{2(1-D_1)} \frac{\partial \xi}{\partial x_3} \right\} \\ &+ \left[(\mathbf{E}_0)_{x_2} \frac{abcB'(\xi)x_2}{2(1-D_2)} + (\mathbf{E}_0)_{x_1} \frac{abcA'(\xi)x_1}{2(1-D_1)} \right] \frac{\partial \xi}{\partial x_3}. \end{aligned} \qquad (17)$$

Clark et al. (1986) give explicit expressions for the demagnetising factors, the functions $A(\xi)$, $B(\xi)$, $C(\xi)$ and their derivatives with respect to ξ, and the derivatives $\partial\xi/\partial x_i$.

The volume of the cavity $\tau = 4\pi abc/3$. If the components of the unperturbed current flow with respect to the ellipsoid axes are j_1, j_2, j_3 the corresponding anomalous magnetic field is:

$$B_1'(\xi < 0) = \frac{\mu_0 j_3 abcB(0)}{2(1-D_3)} x_2 - \frac{\mu_0 j_2 abcC(0)}{2(1-D_2)} x_3 = \mu_0 \left(\frac{j_3 D_2 x_2}{(1-D_3)} - \frac{j_2 D_3 x_3}{(1-D_2)} \right), \qquad (18)$$

$$B_2'(\xi < 0) = \frac{\mu_0 j_1 abcC(0)}{2(1-D_1)} x_3 - \frac{\mu_0 j_3 abcA(0)}{2(1-D_3)} x_1 = \mu_0 \left(\frac{j_1 D_3 x_3}{(1-D_1)} - \frac{j_3 D_1 x_1}{(1-D_3)} \right), \qquad (19)$$

$$B_3'(\xi < 0) = \frac{\mu_0 j_2 abcA(0)}{2(1-D_2)} x_1 - \frac{\mu_0 j_1 abcB(0)}{2(1-D_1)} x_2 = \mu_0 \left(\frac{j_2 D_1 x_1}{(1-D_2)} - \frac{j_1 D_2 x_2}{(1-D_1)} \right), \qquad (20)$$

$$B_1'(\xi > 0) = \frac{\mu_0 j_3 abcB(\xi)}{2(1-D_3)} x_2 - \frac{\mu_0 j_2 abcC(\xi)}{2(1-D_2)} x_3 = \frac{3\mu_0 \tau}{8\pi} \left(\frac{B(\xi) j_3 x_2}{(1-D_3)} - \frac{C(\xi) j_2 x_3}{(1-D_2)} \right), \qquad (21)$$

$$B_2'(\xi > 0) = \frac{\mu_0 j_1 abcC(\xi)}{2(1-D_1)} x_3 - \frac{\mu_0 j_3 abcA(\xi)}{2(1-D_3)} x_1 = \frac{3\mu_0 \tau}{8\pi} \left(\frac{C(\xi) j_1 x_3}{(1-D_1)} - \frac{A(\xi) j_3 x_1}{(1-D_3)} \right), \tag{22}$$

$$B_3'(\xi > 0) = \frac{\mu_0 j_2 abcA(\xi)}{2(1-D_2)} x_1 - \frac{\mu_0 j_1 abcB(\xi)}{2(1-D_1)} x_2 = \frac{3\mu_0 \tau}{8\pi} \left(\frac{A(\xi) j_2 x_1}{(1-D_2)} - \frac{B(\xi) j_1 x_2}{(1-D_1)} \right). \tag{23}$$

The magnetic field within the ellipsoidal cavity is non-uniform, but has a uniform gradient. At the centre of the ellipsoidal cavity ($x_1 = x_2 = x_3 = 0$) the magnetic field is equal to the field that existed at that point before insertion of the cavity. The *resultant* internal magnetic gradient tensor is symmetric and traceless, as required. It is given by:

$$\mathbf{G} = \begin{bmatrix} B_{11}^0 & B_{21}^0 + \mu_0 D_2 j_3/(1-D_3) & B_{13}^0 + \mu_0 D_1 j_2/(1-D_2) \\ B_{21}^0 + \mu_0 D_2 j_3/(1-D_3) & B_{22}^0 & B_{32}^0 + \mu_0 D_3 j_1/(1-D_1) \\ B_{13}^0 + \mu_0 D_1 j_2/(1-D_2) & B_{32}^0 + \mu_0 D_3 j_1/(1-D_1) & B_{33}^0 \end{bmatrix}, \tag{24}$$

The values of the resultant gradient tensor elements depend on $\mathbf{G}_0 = [\, B_{ij}^0 \,]$, which in turn depends on the configuration of the unperturbed current flow.

Application to removal of noise due to ocean swells

Particularly in shallow seas, electric and magnetic fields generated by the magneto-hydrodynamic action of ocean swells can significantly contaminate low frequency marine electromagnetic measurements. The theory of Weaver (1965), which has been well supported by observations, can be used to calculate the electromagnetic effects of swells. Consider gravity wave propagation at frequency f along the $+x$ direction , which produces slowly oscillating currents parallel to the y axis. The components of the oscillating wave-induced field, $\mathbf{B}_\mathrm{w}$, in the sea water, prior to insertion of the measurement capsule, are related to magnetic gradient tensor elements measured within a spheroidal insulating capsule with a vertical symmetry axis by (Clark, 2009b):

$$\left.\begin{aligned} (\mathbf{B}_\mathrm{w})_x(f) &= \frac{igB_{xx}(f)}{4\pi^2 f^2}, \\ (\mathbf{B}_\mathrm{w})_z(f) &= \frac{igB_{xz}(f)}{4\pi^2 f^2}\left[\frac{8\pi^2 f^2 z/g + 1}{8\pi^2 f^2 z/g + 1 - 4G}\right] \end{aligned}\right\} \tag{25}$$

where $i = \sqrt{(-1)}$, $g = 9.8\ \mathrm{ms}^{-2}$, z is the depth and $G = D_1/(1-D_1)$ is a geometric factor related to the demagnetising factor D_1 along a horizontal axis.

When other sources of magnetic field produce negligible gradients, this method effectively isolates the oceanographic magnetic noise and allows it to be removed from the measured magnetic fields using Fourier analysis (Clark, 2009b). Measured electric fields can also be corrected for wave-motion noise, by removing the swell noise $\mathbf{e} = e_y \hat{\mathbf{y}}$ given by:

$$e_y = 2\pi f (\mathbf{B}_\mathrm{w})_z / g. \tag{26}$$

New methods for determination of dipole location and moment vector

Another important application of magnetic sensors is the detection, location and classification (DLC) of magnetic objects, such as naval mines, UXO, shipwrecks, and archaeological artefacts. Apart from their uses in systematic magnetic surveys, gradient tensor measurements have a specific application to manoeuvrable search platforms that home onto compact magnetic targets (Wiegert and Oeschger, 2005, 2006; Wiegert et al., 2007). Compact magnetic bodies can be well represented by a point dipole source, except very close to the body. A number of methods have been proposed for locating dipole targets from magnetic gradient tensor data (e.g. Wynn et al. 1975; Wilson, 1985; Wynn, 1995, 1997). Methods based on point-by-point analysis of the eigenvectors of the tensor tend to be adversely affected by noise in individual measurements of the gradient tensor elements. Furthermore, there is an inherent four-fold ambiguity in obtaining solutions for dipole location and orientation of its moment from point-by-point analysis of gradient tensors (Wynn et al., 1975; Wilson, 1985), which must be resolved by comparing solutions from different sensor locations, rejecting those that are not consistent (the so-called "ghost" solutions) and retaining the solutions that exhibit the best clustering. Existing methods of dipole tracking are also not robust to the contamination of the measured signal by variable background gradients, interfering anomalies, instrument drift or departures of the target from a pure dipole source.

Nara et al. (2006) have presented a neat solution to the single point dipole location problem that uses measurements of the anomalous field vector and gradient tensor, if accurate values of both are available. Along a fixed direction $\hat{\mathbf{r}}$, the field vector **b** is equal to a geometric factor, depending only on the magnitude and orientation of **m**, divided by r^3. Using this fact it can be shown that the displacement vector from the dipole to the measurement point *independent* of the orientation of **m**, is given by:

$$\mathbf{r} = -3\mathbf{G}^{-1}\mathbf{b}, \tag{27}$$

even though each tensor element and vector component on the RHS of this expression depends on $\hat{\mathbf{m}}$. Equation (27) is applicable provided det**G** is nonzero, so the matrix representation of the tensor is invertible. Although Nara et al. (2006) did not treat this aspect, once the location of the dipole is known, determination of the moment becomes a straightforward linear inversion problem. If the anomalous field vector **b** is known to sufficient accuracy, the moment $\mathbf{m} = m(L, M, N) = (m_x, m_y, m_z)$ can be calculated as (Lima et al., 2006):

$$\mathbf{m} = \frac{r^3}{C}\left[\frac{3\mathbf{b.r}}{2r^2}\mathbf{r} - \mathbf{b}\right]. \tag{28}$$

Similarly, given the location $\mathbf{r} = r(n_1, n_2, n_3)$ of the dipole, the expressions for its gradient tensor elements can be rewritten as

$$\boldsymbol{\rho} = \begin{bmatrix} r^4 B_{xx} \\ r^4 B_{xy} \\ r^4 B_{xz} \\ r^4 B_{yy} \\ r^4 B_{yz} \end{bmatrix} = 3C \begin{bmatrix} 3n_1 - 5n_1^3 & n_2 - 5n_1^2 n_2 & n_3 - 5n_1^2 n_3 \\ n_2 - 5n_1^2 n_2 & n_1 - 5n_1 n_2^2 & -5n_1 n_2 n_3 \\ n_3 - 5n_1^2 n_3 & -5n_1 n_2 n_3 & n_1 - 5n_1 n_3^2 \\ n_1 - 5n_1 n_2^2 & 3n_2 - 5n_2^3 & n_3 - 5n_2^2 n_3 \\ -5n_1 n_2 n_3 & n_3 - 5n_2^2 n_3 & n_2 - 5n_2 n_3^2 \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \mathbf{N} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}, \tag{29}$$

where the LHS and the matrix **N** contain only known quantities. This overdetermined matrix equation can be solved in a least squares sense for the components of the moment in terms of these known parameters:

$$\mathbf{m} = \mathbf{N}^{+}\boldsymbol{\rho} = \left(\mathbf{N}^{\mathrm{T}}\mathbf{N}\right)^{-1}\mathbf{N}^{\mathrm{T}}\boldsymbol{\rho}. \tag{30}$$

The method of Nara et al. (2006) can be extended to uniquely determining the dipole location and moment vector from the gradient tensor and the second order gradient (which is a third rank tensor B_{ijk}) at a measurement point. Along a fixed direction $\hat{\mathbf{r}}$, B_{ij} is equal to a geometric factor, depending only on the magnitude and orientation of **m**, divided by r^4. From this it is easily shown that:

$$(\mathbf{r}.\nabla)B_{ij} = -4B_{ij}, \tag{31}$$

which gives an invertible linear relationship between **r**, and first and second order gradient tensor elements. The system of linear equations is overdetermined, so only a subset of the second order gradients is needed to obtain a unique location. For example, if the gradient tensor is measured along a profile segment, parallel to the *x* axis, the dipole location can be calculated directly from the tensor and its along-profile derivative, which can be calculated by numerical differentiation:

$$\mathbf{r} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ -h \end{bmatrix} = -4\begin{bmatrix} \frac{\partial B_{xx}}{\partial x} & \frac{\partial B_{xy}}{\partial x} & \frac{\partial B_{xz}}{\partial x} \\ \frac{\partial B_{xy}}{\partial x} & \frac{\partial B_{yy}}{\partial x} & \frac{\partial B_{yz}}{\partial x} \\ \frac{\partial B_{xz}}{\partial x} & \frac{\partial B_{yz}}{\partial x} & \frac{\partial B_{zz}}{\partial x} \end{bmatrix}^{-1} \begin{bmatrix} B_{xx} \\ B_{xy} \\ B_{xz} \end{bmatrix}. \tag{32}$$

The moment of the dipole can then be determined from **r** and the gradient tensor as shown above. Thus the dipole location and moment can be found from the first and second order gradient tensors at a single point. This result was inferred empirically by Wynn (1995) and later proved explicitly by him (Wynn, 1997). However that proof does not provide a method for determining the location and moment vector of the dipole. Wynn's (1995) method for inverting the gradient tensor and its along-profile gradients is quite complicated and involves a computationally intensive numerical search algorithm.

The second method presented here analyses data collected along a profile that passes near a dipole target. Unlike most other gradient tensor inversion techniques, this method can correct for contamination of the dipole signature by geological gradients or instrumental drifts, for example. Full details are given in Clark (2008).

Wilson (1985) showed that the scaled moment μ of a dipole, which is a particularly useful rotational invariant because it is independent of magnetic moment orientation and always peaks at the closest point of approach, can be calculated directly from the eigenvalues λ_i of the tensor. A sequence of calculated scaled moments along a profile can be deconvolved and interference terms estimated and removed, but in practice it is easier and to process a related quantity, which yields more robust solutions. Define another invariant that is independent of the dipole orientation by $\nu = \sqrt{(\mu/3)} = \{[\sqrt{(-\lambda_2^2 - \lambda_1\lambda_3)}]/3\}^{½}$, where λ_2 is the eigenvalue with the smallest absolute value. For a pure dipole signature ν is proportional to $\sqrt{m}/r^2$. Then at any point around an isolated dipole source ν can be estimated from the measured eigenvalues. In the presence of background gradients or interference from neighbouring bodies, at successive points $x = x_i$ (i = 1, 2,..., n) along a straight and level path, defined by $y - y_0$ = Y, ν determined from the measured data can be modelled as:

$$\nu_i = \frac{\sqrt{Cm}}{(x_i - x_0)^2 + S^2} + \mathrm{a} + \mathrm{b}x_i + \mathrm{c}x_i^2, \qquad (33)$$

where C is a constant that depends on the system of units, $S = \sqrt{(Y^2 + h^2)}$ is the slant distance from the point of closest approach to the dipole, $x = x_0$ is the point of closest approach, h is the depth of the dipole, a is the base level, and b, c are linear and quadratic terms that represent interference from other anomalies. The deconvolution problem is to solve for the unknown parameters x_0, S, m, a, b, c. This is equivalent to conventional Werner deconvolution (e.g. Ku and Sharp, 1983) of the TMI anomaly of a thin sheet.

Once the origin of x and slant distance are determined and the scaled moment, μ_i, and distance to source, r_i, at successive points are known, the measured gradient tensor elements can be modelled by:

$$B_{xx}^{(i)} = -\frac{\mu_i}{r_i^3}\left[2Lx_i^3 + 4M'Sx_i^2 - 3LS^2x_i - M'S^3\right] + \mathrm{a}_{\mathrm{xx}} + \mathrm{b}_{\mathrm{xx}}x_i + \mathrm{c}_{\mathrm{xx}}x_i^2, \qquad (34)$$

with similar terms for the other four independent tensor elements, where the distances along the profile, x_i, are now with respect to an origin at the point of closest approach, $M' = MY - Nh$ is the direction cosine of the slant component of magnetization and a quadratic interference term is assumed for each component. The deconvolution problem is to solve for the unknown parameters L, M, N, Y, h and the interference terms a_{ij}, b_{ij}, c_{ij}. This is carried out in a similar way to the deconvolution of the invariant ν (Clark, 2008). At this stage it is recommended to remove the interference terms from the measured tensor elements and recalculate the eigen-values and ν. Using the new estimates of x_0, S, m the deconvolution of the tensor elements can be repeated. The process is generally rapidly convergent, the revised interference terms become small and the source parameters become more precisely determined.

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