Two Examples of Submodularity in Wireless Communications

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Submodularity

- a property of functions defined on lattice [1][1] [2][2].
  - lattice: fundamental algebraic structure on partial order.
- applications: economics, machine learning, operations research.

Study on Machine Learning in [3][3]:

Submodularity imposes a structure which allows much stronger mathematical results than we would be able to achieve without it.

- submodularity on
  - vector lattice: discrete convexity and comparative statics
  - set lattice: combinatorial optimization, e.g., in graph theory

[1] Topkis 2001
[2] Murota 2005
[3] Vondrak 2007

Lattice

Poset

For a set $\mathcal{L}$ and a binary order $\preceq$, $(\mathcal{L}, \preceq)$ is a *poset (partially ordered set)* if either $a \preceq b$ or $a \not\preceq b, \forall a, b \in \mathcal{L}$.

examples: $(\mathbb{R}^N, \leq)$, $(\{1, \ldots, 4\}^N, \leq)$, $(2^V, \subseteq)$ and $(\{\{1\}, \{1, 2\}, \{3\}\}, \subseteq)$

Lattice

A poset $(\mathcal{L}, \preceq)$ is a *lattice* with notation $(\mathcal{L}, \vee, \wedge)$ if $a \vee b = \sup\{a, b\} \in \mathcal{L}$ and $a \wedge b = \inf\{a, b\} \in \mathcal{L}, \forall a, b \in \mathcal{L}$ with sup and inf w.r.t. $\preceq$

- maximum $\bigvee \mathcal{L} = \sup \mathcal{L}$ and minimum $\bigwedge \mathcal{L} = \inf \mathcal{L}$ exist;

examples: $(\mathbb{R}^N, \vee, \wedge)$, $(\{1, \ldots, 4\}^N, \vee, \wedge)$ with $\mathbf{r} \vee \mathbf{r}' = (\max\{r_i, r_i'\} \colon i \in \{1, \ldots, N\})$ and $\mathbf{r} \wedge \mathbf{r}' = (\min\{r_i, r_i'\} \colon i \in \{1, \ldots, N\})$; $(2^V, \cup, \cap)$ and $(\{\{1\}, \{1, 2\}\}, \cup, \cap)$

Submodular Function

Submodularity

$f : (\mathcal{L}, \vee, \wedge) \mapsto \mathbb{R}$ is *submodular* if

$$f(a) + f(b) \geq f(a \vee b) + f(a \wedge b), \quad \forall a, b \in \mathcal{L}.$$

f is supermodular if $-f$ is submodular.

Tarski Fixed Point Theorem

N-player game model $\Omega = \{\mathcal{N}, \{\mathcal{A}_i, c_i(\mathbf{a})\}_{i \in \mathcal{N}}\}$ with $\mathbf{a} \in \mathcal{A} = \times_{i \in \mathcal{N}} \mathcal{A}_i \subseteq \mathbb{R}^N$:

- pure strategy Nash equilibrium (PSNE): *best response function* $\boldsymbol{\psi} \colon \mathcal{A} \mapsto \mathcal{A}$ with $a_i \in \mathcal{A}_i$, $\mathbf{a}_{-i} \in \times_{i' \in \mathcal{N} \setminus \{i\}} \mathcal{A}_{i'}$ and

$$\psi_i(a_{-i}) \in \arg\min \left\{ c_i(a_i, \mathbf{a}_{-i}) \colon a_i \in \mathcal{A}_i \right\}, \qquad \forall i \in \mathcal{N}.$$

  $\mathbf{a}^*$ is an PSNE if $\mathbf{a}^* = \boldsymbol{\psi}(\mathbf{a}^*)$, i.e., $\mathbf{a}^*$ is a fixed point of $\boldsymbol{\psi}$.

- question: PSNE exists for discrete $\mathcal{A}$? supermodular game with strategic complements: $\boldsymbol{\psi} \colon (\mathcal{A}, \vee, \wedge) \mapsto (\mathcal{A}, \vee, \wedge)$ is non-decreasing if c_i is submodolar $\forall i$.

Tarski Fixed Point Theorem [4][4]

The fixed points of non-decreasing $\boldsymbol{\psi} \colon (\mathcal{A}, \vee, \wedge) \mapsto (\mathcal{A}, \vee, \wedge)$ form a (nonempty) lattice.

[4] Tarski *et. al* 1955

Submodular (Set) Function Minimization

For $f\colon (2^V, \cup, \cap) \mapsto \mathbb{R}$, consider

$$\min\{f(X)\colon X \subseteq V\} \tag{1}$$

combinatorial optimization: NP-complete or NP-hard in general

SFM (submodular function minimization) algorithm

If f is submodular, i.e.,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \qquad \forall X, Y \subseteq V,$$

(1) can be solved in polynomial time and the minimizers form a lattice: $\bigcup \operatorname{argmin}\{f(X)\colon X \subseteq V\}$ and $\bigcap \operatorname{argmin}\{f(X)\colon X \subseteq V\}$ exist [5][5].

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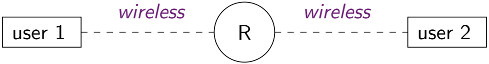
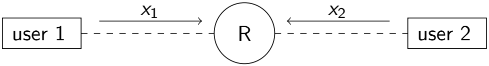
[5] Fujishige 2005

Adaptive Modulation in Network-coded Two-way Relay Channel

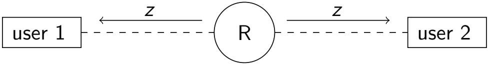
network-coded two-way relay channel (NC-TWRC): two users communicate via a center node, relay 'R'.



physical-layer network coding (PNC): messages x_1 and x_2 transmitted simultaneously in phase I, the superposition z broadcast in phase II.



phase I: multiple access (MAC)



phase II: amplify and forward (AF)

Adaptive Modulation in NC-TWRC

assumption: m-quadrature amplitude modulation (m-QAM) adopted by each user:

- constellation size $m_i = 2^{a_i}$ of user i with $a_i \in \{0, 1, \dots\}$, the number of bits/symbol, determined by user i

Strategic Complements

increasing best response:

- spectral efficiency: one tends to transmit while the other does so
- equal share of the channel: one increases a_i while the other $-i = \{1, 2\} \setminus \{i\}$ does so

proposal: two-player game model parameterized by user-to-user channel signal-to-noise ratios (SNRs)

Two-player Game Model

$\Omega_\gamma = \{\mathcal{N}, \Gamma, \{\mathcal{A}_i, c_i(\gamma_i, \mathbf{a})\}_{i\in\mathcal{N}}\}$:

- $\mathcal{N} = \{1, 2\}$;
- $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \Gamma = \Gamma_1 \times \Gamma_2$ with $\gamma_i \in \Gamma_i$ being SNR of user i-to-user $-i$ channel determined by PNC scheme;
- $\mathbf{a} = (a_1, a_2) \in \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = \{0, 1, \dots, A_m\}^2$ with A_m being the maximum number of bits/symbol
- cost function: $c_i : \Gamma_i \times \mathcal{A} \mapsto \mathbb{R}_+$

$$c_i(\gamma_i, \mathbf{a}) = c_e(\gamma_i, a_i) + c_r(\mathbf{a})$$

with the cost associated with transmission error rate

$$c_e(\gamma_i, a_i) = \frac{-\ln(5\bar{P}_b)(2^{a_i} - 1)}{1.5\gamma_i}$$

and the cost associated with spectral efficiency and fairness

$$c_r(\mathbf{a}) = \frac{a_{-i} + 1}{a_i + 1}$$

Pure Strategy Nash Equilibrium (PSNE)

Submodularity

$c_i\colon (\mathcal{A}, \vee, \wedge) \mapsto \mathbb{R}_+$ is submodular, i.e.,

$$c_i(\gamma_i, \mathbf{a}) + c_i(\gamma_i, \mathbf{a}') \geq c_i(\gamma_i, \mathbf{a} \vee \mathbf{a}') + c_i(\gamma_i, \mathbf{a} \wedge \mathbf{a}'),$$

for all $\mathbf{a}, \mathbf{a}' \in (\mathcal{A}, \vee, \wedge)$, $i \in \mathcal{N}$ and $\boldsymbol{\gamma} \in \Gamma$.

Tarski Fixed Point Theorem $\implies$ Existence of PSNE

Pure strategy $\boldsymbol{\theta}\colon \Gamma \mapsto \mathcal{A}$, where $\boldsymbol{\theta}(\boldsymbol{\gamma}) = (\theta_1(\boldsymbol{\gamma}), \theta_2(\boldsymbol{\gamma}))$ with $\theta_i(\boldsymbol{\gamma}) \in \mathcal{A}_i$ being the pure strategy of user i when SNRs are $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$:

- PSNE $\boldsymbol{\theta}^*$ exists
- The largest PSNE $\overline{\boldsymbol{\theta}}^*$ and the smallest PSNEs $\underline{\boldsymbol{\theta}}^*$ exist

Cournot Tatonnement

determine extremal PSNEs: Cournot tatonnement [6][6]

- Let $\overline{\boldsymbol{\psi}} \colon \Gamma \times \mathcal{A} \mapsto \mathcal{A}$ and $\underline{\boldsymbol{\psi}} \colon \Gamma \times \mathcal{A} \mapsto \mathcal{A}$ be the maximal and minimal best response functions, respectively, with

$$\overline{\psi}_i(\boldsymbol{\gamma}, a_{-i}) = \bigvee \arg \min_{a_i \in \mathcal{A}_i} \; c_i(\gamma_i, a_i, a_{-i})$$

$$\underline{\psi}_i(\boldsymbol{\gamma}, a_{-i}) = \bigwedge \arg \min_{a_i \in \mathcal{A}_i} \; c_i(\gamma_i, a_i, a_{-i})$$

- recursions with $\overline{\boldsymbol{\theta}}^{(0)}(\boldsymbol{\gamma}) = \bigvee \mathcal{A}$ and $\underline{\boldsymbol{\theta}}^{(0)}(\boldsymbol{\gamma}) = \bigwedge \mathcal{A}$:

$$\overline{\boldsymbol{\theta}}(\boldsymbol{\gamma}) \coloneqq \overline{\boldsymbol{\psi}}(\boldsymbol{\gamma}, \overline{\boldsymbol{\theta}}(\boldsymbol{\gamma}))$$

$$\underline{\boldsymbol{\theta}}(\boldsymbol{\gamma}) \coloneqq \underline{\boldsymbol{\psi}}(\boldsymbol{\gamma}, \underline{\boldsymbol{\theta}}(\boldsymbol{\gamma}))$$

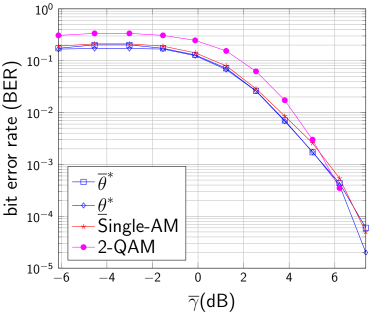
Convergence

$\{\overline{\boldsymbol{\theta}}^{(k)}(\boldsymbol{\gamma})\}$ and $\{\underline{\boldsymbol{\theta}}^{(k)}(\boldsymbol{\gamma})\}$ converge monotonically downward and upward to $\overline{\boldsymbol{\theta}}^{*}(\boldsymbol{\gamma})$ and $\underline{\boldsymbol{\theta}}^{*}(\boldsymbol{\gamma})$, respectively, for all $\boldsymbol{\gamma}$.

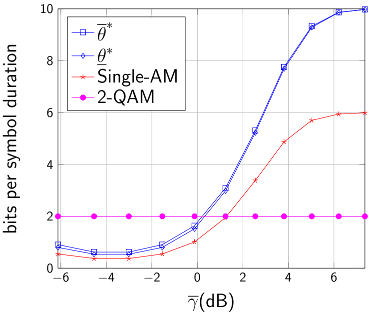
[6] Vives 1990

Experiment I

performance of extremal PSNEs:
$\mathcal{A} = \{0, 1, \ldots, 9\}^2$ and simulation lasts for 10^4 symbol durations: extremal PSNEs $\overline{\boldsymbol{\theta}}^*$ and $\underline{\boldsymbol{\theta}}^*$ are compared to the single-agent adaptive modulation (Single-AM) and 2-QAM scheme.



(a) bit error rate (BER)

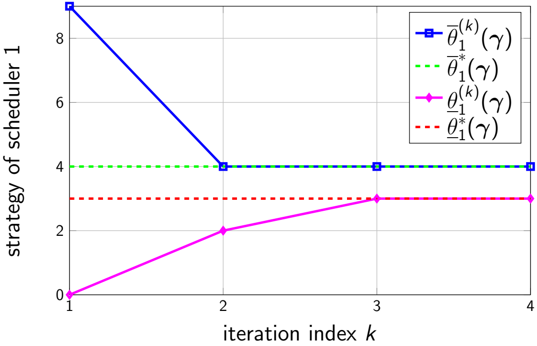


(b) spectral efficiency

Experiment II

example of Cournot tatonnement

$\mathcal{A} = \{0, 1, \ldots, 9\}^2$ and the sequences $\{\overline{\theta}_1^{(k)}(\gamma)\}$ and $\{\underline{\theta}_1^{(k)}(\gamma)\}$ generated by Cournot tatonnement for certain γ



Communication for Omniscience

indices of users: a finite *ground set* V with $|V| > 1$
discrete correlated random source: $Z_V = (Z_i : i \in V)$

- user i observes an i.i.d. n-sequence Z_i^n of Z_i in private

communication for omniscience (CO) [7][7]:

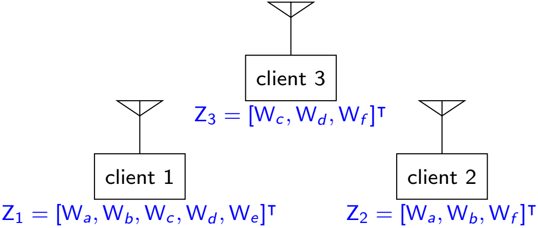
- users exchange Z_is directly over noiseless broadcast channels
- goal: attain omniscience, the state that each user recovers Z_V^n

Minimum Sum-rate Problem

how to attain omniscience with $R_{CO}(V)$, the *minimum total number of transmissions*: value of $R_{CO}(V)$ and an optimal rate vector $\mathbf{r}_V^* = (r_i^* : i \in V)$

[7] Csiszar *et. al* 2004: CO formulated based on the study on secret capacity

Example: Coded Cooperative Data Exchange (CCDE)



3-mobile clients in $V = \{1, 2, 3\}$; Z_i: partial observation of a packet set with W_j denoting a packet

Solutions to Minimum Sum-rate Problem

$R_{CO}(V) = \frac{7}{2}$ and $\mathbf{r}_V^* = (r_1^*, r_2^*, r_3^*) = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$: by packet-splitting $W_j \Longrightarrow W_j^{(1)}, W_j^{(2)}$; transmit $(r_1, r_2, r_3) = (5, 1, 1)$ with r_i denote the number of linear combinations of packet chunks $W_j^{(k)}$.

Omniscience-achievability

For $X \subseteq V$: $r(X) = \sum_{i \in X} r_i$ for $\mathbf{r}_V = (r_i : i \in V)$
$H(X)$: the amount of randomness in Z_X measured by Shannon entropy

Omniscience-achievability [7][8]

An omniscience-achievable $\mathbf{r}_V$ satisfies the Slepian-Wolf (SW) constraint:
$r(X) \geq H(X|V \setminus X) = H(V) - H(V \setminus X), \forall X \subsetneq V$.

achievable rate vector set:

$$\mathcal{R}(V) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} \colon r(X) \geq H(X|V \setminus X), \forall X \subsetneq V\}$$

minimum sum-rate:

$$R_{\text{CO}}(V) = \min\{r(V) \colon \mathbf{r}_V \in \mathcal{R}(V)\}$$

constant sum-rate set: $\mathcal{R}_\alpha(V) = \{\mathbf{r}_V \in \mathcal{R}(V) \colon r(V) = \alpha\}$
optimal rate vector set: $\mathcal{R}_{R_{\text{CO}}(V)}(V)$

[8] Csiszár *et. al* 2004

Nonemptiness of Base Polyhedron

For $\alpha \in \mathbb{R}_+$, let

$$f_\alpha(X) = \begin{cases} H(X|V \setminus X) & X \subsetneq V \\ \alpha & X = V \end{cases}.$$

polyhedron: $P(f_\alpha, \geq) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} \colon r(X) \geq f_\alpha(X), \forall X \subseteq V\}$
base polyhedron: $B(f_\alpha, \geq) = \{\mathbf{r}_V \in P(f_\alpha, \geq) \colon r(V) = f_\alpha(V) = \alpha\}$

- $B(f_\alpha, \geq) = \mathcal{R}_\alpha(V) \neq \emptyset \iff \exists$ achievable $\mathbf{r}_V$ with $r(V) = \alpha$

dual set function: $f_\alpha^\#(X) = f_\alpha(V) - f_\alpha(V \setminus X)$

- $B(f_\alpha, \geq) = B(f_\alpha^\#, \leq)$ [5][9];

why consider $B(f_\alpha^\#, \leq)$? $f_\alpha^\#$ is *intersecting submodular*, i.e.,

$$f_\alpha^\#(X) + f_\alpha^\#(Y) \geq f_\alpha^\#(X \cup Y) + f_\alpha^\#(X \cap Y), \quad \forall X, Y : X \cap Y \neq \emptyset.$$

[9] Fujishige 2005

Minimum Sum-rate

$\Pi(V)$: the set of all partitions of V and $\Pi'(V) = \Pi(V) \setminus \{V\}$
achievability of α: $B(f_\alpha^{\#}, \leq) \neq \emptyset$ iff $\alpha = \min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_\alpha^{\#}(C)$ [5][10]

Minimum Sum-rate

$R_{\text{CO}}(V) = \max_{\mathcal{P} \in \Pi'(V)} \phi(\mathcal{P})$ with the finest maximizer $\mathcal{P}^*$. Here,

$$\phi(\mathcal{P}) = \sum_{C \in \mathcal{P}} \frac{H(V \setminus C | C)}{|\mathcal{P}| - 1}.$$

interpretation: $\forall C \in \mathcal{P}$, the cut $\{C, V \setminus C\}$ imposes SW constraint $r(V \setminus C) \geq H(V \setminus C | C)$ so that

$$\sum_{C \in \mathcal{P}} r(V \setminus C) = (|\mathcal{P}| - 1) r(V) \geq \sum_{C \in \mathcal{P}} H(V \setminus C | C)$$

A multi-way cut $\mathcal{P} \in \Pi'(V)$ imposes $r(V) \geq \phi(\mathcal{P})$.

[10] Fujishige 2005: $\min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_\alpha^{\#}(C)$ is called the *Dilworth truncation* of $f_\alpha^{\#}$.

Principal Sequence of Partitions

Principal Sequence of Partitions (PSP) [8][11]:

$\min_{\mathcal{P}\in\Pi(V)} \sum_{C\in\mathcal{P}} f_{\alpha}^{\#}(C)$ is a piecewise linear increasing curve in α that is fully characterized by $p \leq |V|-1$ critical points

$$H(V) = \alpha_0 > \alpha_1 > \alpha_2 > \ldots > \alpha_p \geq 0.$$

Let $\mathcal{P}_j$ be the finest minimizer of $\min_{\mathcal{P}\in\Pi(V)} \sum_{C\in\mathcal{P}} f_{\alpha}^{\#}(C)$.

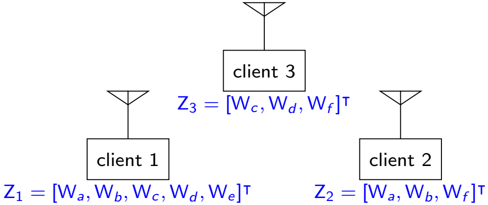
$$\mathcal{P}_0 \succ \mathcal{P}_1 \succ \mathcal{P}_2 \succ \ldots \succ \mathcal{P}_p$$

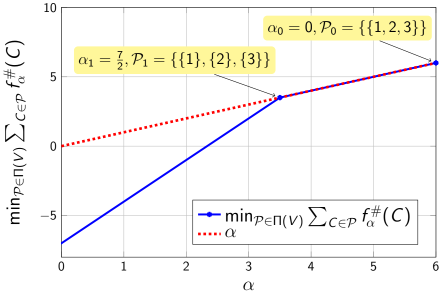
where $\mathcal{P} \succ \mathcal{P}'$ denotes $\mathcal{P}'$ is strictly finer than $\mathcal{P}$.

The first critical point determines the solutions to the minimum sum-rate problem: $R_{\text{CO}}(V) = \alpha_1, \mathcal{P}^* = \mathcal{P}_1$.

[11] Nagano *et. al* 2010

Example of PSP





PSP results:

$\alpha_0 > \alpha_1$ and $\mathcal{P}_0 \succ \mathcal{P}_1$:

- No omniscience-achievable $\mathbf{r}_V$ if $\alpha < \alpha_1$, because $\alpha \neq \min_{\mathcal{P}\in\Pi(V)} \sum_{C\in\mathcal{P}} f_\alpha^\#(C)$;
- $R_{\text{CO}}(V) = \alpha_1$ and $\mathcal{P}^* = \{\{1\},\{2\},\{3\}\} = \mathcal{P}_1$

Properties of $\phi(\mathcal{P})$ in PSP

α_j and $\mathcal{P}_j$:

- If $j = 1$, $\alpha_j = \phi(\mathcal{P}_j)$;
- When $j > 1$, let $\alpha = \phi(\mathcal{P}_j)$ and $\mathcal{P}_{j'}$ be the finest minimizer of $\min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_\alpha^{\#}(C)$. Then,

$$\alpha_j < \alpha < \alpha_1$$

$$j' < j \Longrightarrow \mathcal{P}_{j'} \succ \mathcal{P}_j$$

Suggestion: A Recursive Algorithm

- iteratively updates α and $\mathcal{P}$, the estimation of $\alpha_1 = R_{\text{CO}}(V)$ and $\mathcal{P}^* = \mathcal{P}_1$;
- terminate when $\alpha = \phi(\mathcal{P})$.

Modified Decomposition Algorithm

recursion in modified decomposition algorithm (MDA):

$$\alpha \coloneqq \phi(\mathcal{P})$$

$\mathcal{P}^{(n)}$ is the finest minimizer of $\min_{\mathcal{P}\in\Pi(V)} \sum_{C\in\mathcal{P}} f^{\#}_{\alpha^{(n)}}(C)$ and $\mathcal{P}^{(0)} = \{\{i\} \colon i \in V\}$

Optimality of the MDA algorithm

$\{\alpha^{(n)}\}$ and $\{\mathcal{P}^{(n)}\}$ converge monotonically towards $\alpha_1 = R_{\text{CO}}(V)$ and $\mathcal{P}_1 = \mathcal{P}^*$, respectively. Also returns $\mathbf{r}_V \in B(f^{\#}_{R_{\text{CO}}(V)}, \leq) = \mathfrak{R}_{R_{\text{CO}}(V)}(V)$

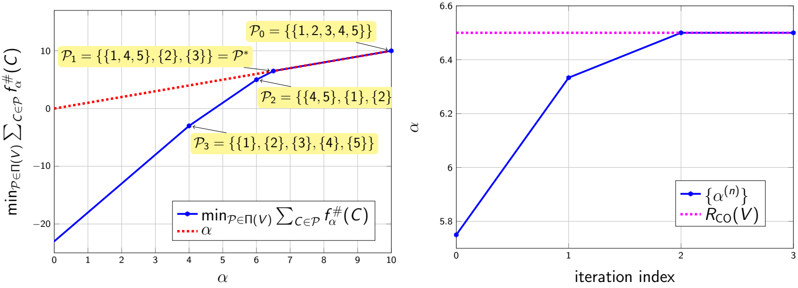
- $\min_{\mathcal{P}\in\Pi(V)} \sum_{C\in\mathcal{P}} f^{\#}_{\alpha^{(n)}}(C)$ reduces to $\bigcap \text{argmin}\{f^{\#}_{\alpha^{(n)}}(X) - r(X) \colon i \in X \subseteq V\}, \forall i \in V$, SFM due to the intersecting submodularity of $f^{\#}_{\alpha^{(n)}}$
- complexity: $O(|V|^2 \cdot \text{SFM}(|V|))$[12]

[12] SFM(|V|): the complexity of minimizing submodular function $f : 2^V \mapsto \mathbb{R}$.

Experiment

$V = \{1, \dots, 5\}$: W_m is an independent uniformly distributed random bit:

$$\begin{aligned} \mathsf{Z}_1 &= (\mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_d, \mathsf{W}_h, \mathsf{W}_i), \\ \mathsf{Z}_2 &= (\mathsf{W}_e, \mathsf{W}_f, \mathsf{W}_h, \mathsf{W}_i), \\ \mathsf{Z}_3 &= (\mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_e, \mathsf{W}_j), \\ \mathsf{Z}_4 &= (\mathsf{W}_a, \mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_d, \mathsf{W}_f, \mathsf{W}_g, \mathsf{W}_i, \mathsf{W}_j), \\ \mathsf{Z}_5 &= (\mathsf{W}_a, \mathsf{W}_b, \mathsf{W}_c, \mathsf{W}_f, \mathsf{W}_i, \mathsf{W}_j), \end{aligned}$$



(c) PSP

(d) $\{\alpha^{(n)}\}$ by MDA algorithm

Extensions of CO: Secret Capacity

secret capacity $\mathcal{C}_S(V)$: the maximum rate at which the secret key can be generated by the users in V with results in [7][13]:

- dual relationship: $R_{\text{CO}}(V) = H(V) - \mathcal{C}_S(V)$
- *mutual dependence* upper bound on $\mathcal{C}_S(V)$:

$$\mathcal{C}_S(V) \leq \underbrace{I(V) = \min_{\mathcal{P} \in \Pi'(V)} \frac{\sum_{C \in \mathcal{P}} H(C) - H(V)}{|\mathcal{P}| - 1}}_{\text{mutual dependence in } Z_V}$$

tightness [9][14]: $\mathcal{C}_S(V) = I(V) = H(V) - R_{\text{CO}}(V)$
question: how to achieve $\mathcal{C}_S(V)$? with interactive communication rate $r(V) = R_{\text{CO}}(V)$? silly! $\mathcal{C}_S(V)$ can be attained with $r(V) \leq R_{\text{CO}}(V)$ [7]

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[13] Csiszár *et. al* 2004

[14] Chan *et. al* 2015

Extensions of CO: Clustering

Inspired by the name 'mutual dependence'

$$I(V) = \min_{\mathcal{P}\in\Pi'(V)} \frac{\sum_{C\in\mathcal{P}} H(C) - H(V)}{|\mathcal{P}| - 1}$$

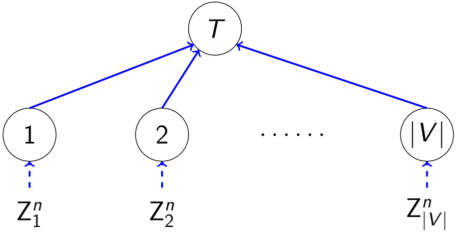
is proposed in [9][15] as a generalization of Shannon's mutual information to multivariate case: $I(V) = H(\{1\}) + H(\{2\}) - H(\{1,2\})$ when $V = \{1,2\}$.

- realization: $I(V)$ is the similarity measure of more than two rvs.
  - limitation in existing clustering algorithms: pairwise similarity/dissimilarity measure
  - agglomerative clustering result given by PSP determined in $O(|V|^2 \cdot \text{SFM}(|V|))$ time

question: can PSP clustering frame work provide more objective overview of the dataset?

[15] Chan *et. al* 2015

Extensions of CO (Digiscape): Source Coding with Side Information



sensor nodes $i \in V = \{1, \ldots, |V|\}$ reveal all information to sink T.

- for lossless data compression/aggregation, SW constraints:

$$r(X) \geq H(X|V \setminus X), \forall X \subsetneq V, r(V) = H(V) \Longrightarrow \mathcal{R}_{H(V)}(V)$$

- an extreme, one of the unfairest, $\mathbf{r}_V \in \mathcal{R}_{H(V)}(V)$ can be determined in $O(|V|)$ time

Conclusion

two examples of submodularity in wireless communications:

- vector lattice: the existence of PSNEs in a game modeled adaptive modulation problem in NC-TWRC
- set lattice: polynomial time algorithm for solving CO problem

future:

- vector lattice: more applications of discrete convexity, e.g., the energy-delay trade-off in data aggregation tree in Digiscape, and monotone comparative statics
- set lattice: improving efficiency for determining PSP
  - less call of SFM algorithm
  - improving complexity SFM($|V|$): SFM belongs to worst polynomial algorithm category, e.g., $|V|^5$ to $|V|^8$

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