On the Estimation of the Parameters of a Real Sinusoid in Noise

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***Abstract*—We propose and comprehensively analyze a computationally efficient algorithm to estimate the parameters of a real sinusoidal signal in noise. This method uses the fast Fourier Transform and is, therefore, computationally efficient. Accounting for the interference due to the negative spectral component allows the frequency to be estimated very accurately. Estimates of the amplitude and phase are derived in the process and are necessary for the suppression of the leakage. Theoretical analysis establishes that the estimator is asymptotically unbiased and achieves the Cramér–Rao lower bound. Simulation results are presented to verify the theory and demonstrate that the estimation performance is superior to other estimators in the literature.**

***Index Terms*—Fourier coefficients, frequency estimation, Interpolation algorithm, real sinusoid.**

I. Introduction

THE estimation of the parameters of a real sinusoidal signal in noise is a classical yet important research problem in many applications [1], [2]. The signal model is

$$x(n) = a\cos(2\pi f_0 n + \phi) + w(n), \quad n = 0, \ldots, N-1 \quad (1)$$

where N is the number of samples, a and ϕ are the amplitude and initial phase, and $f_0 \in (0, 0.5]$ is the frequency. The noise terms $w(n)$ are real Gaussian with zero mean and variance σ^2. Then, the signal-to-noise ratio (SNR) is $\rho = |a|^2/2\sigma^2$.

Much work has been done on the estimation of the parameters f_0, a, and ϕ [3]. Traditional time-domain methods, such as Prony's and Pisarenko's methods, and the MUSIC algorithm [2], are outperformed by later algorithms such as ESPRIT [4] and Matrix Pencil [5]. The recently proposed weighted linear prediction (WLP) approach [6] is capable of attaining the Cramér–Rao bound (CRB) by employing an iterative procedure to minimize the weighted least squares error between the signal model and noisy data. All these methods, however, are computationally complex as they require a matrix inversion, which is $O(N^3)$. Frequency-domain estimators, on the other hand, are mostly based on interpolation using DFT coefficients [1], [7], and [8]. These estimators take advantage of the fast Fourier transform (FFT) and are computationally much simpler than the time-domain approaches. However, they are plagued by estimation bias caused by spectral leakage [9]. Windowing the signal [1], [10] achieves some bias reduction but at the expense of increased estimation variance.

The limitations of the aforementioned methods, are overcome by a novel estimation strategy that employs the FFT and is, therefore, computationally fast [11]. Unlike previous FFT-based attempts, this algorithm is effective, under the standard rectangular window, at eliminating estimation biases caused by spectral leakage. As a result, it achieves accurate unbiased estimates without compromising the estimation variance. The estimator employs an iterative procedure based on an accurate interpolation technique with excellent convergence properties [12]. Thus, interference between the positive and negative frequency components is successfully mitigated by reconstructing the negative component and subtracting it from the signal [13]. In this letter, we elucidate the relationship between our algorithm and a recently proposed method [14], showing that the latter lacks a crucial step which leads to degraded performance when the SNR is sufficiently high or the frequency is small. Although the estimator obtains estimates of all parameters, we focus on the frequency estimates and present theoretical analysis of our method to establish its performance and convergence behavior.

This letter is organized as follows. In Section II, we present the new algorithm and explain its relationship to [14] in Section III. In Section IV, we establish the theoretical performance and demonstrate this performance using simulation results in Section V. Finally, conclusions are drawn in Section VI.

II. New Estimation Algorithm

Noting that the real sinusoid in (1) can be written as the sum of two complex exponentials, $s(n) = ae^{j\phi}e^{j2\pi f_0 n}/2$ and $s^*(n) = ae^{-j\phi}e^{-j2\pi f_0 n}/2$, we propose an algorithm that takes into account the relationship between the two components to efficiently estimate the signal parameters. This two-stage estimator employs an iterative procedure, where in each iteration the leakage due to $s^*(n)$ is effectively suppressed [13].

The detailed algorithm is given in Table I. Let us put $A = ae^{j\phi}/2$. Now the frequency of $s(n)$ can be expressed as

$$f_0 = \frac{m_0 + \delta_0}{N} \quad (2)$$

where $m_0 = [Nf_0]$ and $\delta_0 \in [-0.5, 0.5]$ is a frequency residual. Here, $[\bullet]$ is the rounding operation. It is well established that, above the SNR breakdown threshold, the maximum of the DFT is m_0 almost surely [15]. We refine this coarse estimate by successively estimating the residual δ_0. The fine estimator of [8] has highly desirable convergence properties for a single exponential, but exhibits degraded performance when applied to two components. We address this problem by incorporating a leakage subtraction strategy into the iterations [16].

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TABLE I
PROPOSED ESTIMATOR

Given	A real sinusoid $x(n)$, $n = 0\ldots, N-1$;
Calculate	$X(k) = \text{FFT}(x)$ and $Y(k) = \lvert X(k)\rvert^2$;
Find	$\hat{m}_0 = \arg\max_{0\le k\le N/2} Y(k)$;
Set	$\hat{\delta} = 0$ and $\hat{A} = 0$;
Loop	For i from 1 to Q, do
	(1) $X_p = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}(\hat{m}_0+\hat{\delta}+p)n}$ $p = \pm 0.5$;
	(2) $\hat{L}_p = \frac{\hat{A}^*}{N}\frac{1+e^{-j4\pi\hat{\delta}}}{1-e^{-j\frac{2\pi}{N}(2\hat{m}+2\hat{\delta}+p)}}$, and $\hat{S}_p = X_p - \hat{L}_p$;
	(3) $\hat{\delta} = \hat{\delta} + \frac{1}{2}\text{Re}\left\{\frac{\hat{S}_{0.5}+\hat{S}_{-0.5}}{\hat{S}_{0.5}-\hat{S}_{-0.5}}\right\}$;
	(4) $\hat{A} = \frac{1}{N}\left(\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}(\hat{m}_0+\hat{\delta})n} - \hat{A}^*\frac{1-e^{-j4\pi\hat{\delta}}}{1-e^{-j\frac{4\pi}{N}(\hat{m}_0+\hat{\delta})}}\right)$;
Return	$\hat{a} = \frac{\lvert\hat{A}\rvert}{2}$, $\hat{\phi} = \angle\hat{A}$, and $\hat{f}_0 = \frac{\hat{m}_0+\hat{\delta}}{N}$.

Let $\hat{\delta}^{(i-1)}$ be the estimate of δ_0 obtained in iteration $i-1$. Then, in the ith iteration, the noiseless DFT coefficients at locations $m_0 + \hat{\delta}^{(i-1)} + p$ with $p = \pm 0.5$, denoted by X_p, are

$$
\begin{aligned}
X_p &= \frac{1}{N}\sum_{n=0}^{N-1}[s(n)+s^*(n)]e^{-j\frac{2\pi}{N}(m_0+\hat{\delta}^{(i-1)}+p)n} \\
&= \frac{A}{N}\frac{1+e^{j2\pi(\delta_0-\hat{\delta}^{(i-1)})}}{1-e^{j\frac{2\pi}{N}(\delta_0-\hat{\delta}^{(i-1)}-p)}} + \frac{A^*}{N}\frac{1+e^{-j2\pi(\delta_0+\hat{\delta}^{(i-1)})}}{1-e^{-j\frac{2\pi}{N}(2m_0+\delta_0+\hat{\delta}^{(i-1)}+p)}} \\
&= S_p + L_p
\end{aligned} \tag{3}
$$

where S_p are the Fourier coefficients of $s(n)$ and L_p are the leakage terms due to $s^*(n)$. It is clear that removing the leakage yields $X_p = S_p$ and the estimate δ_0 can be estimated as

$$\hat{\delta}^{(i)} = \hat{\delta}^{(i-1)} + \text{Re}\{h\} \tag{4}$$

where Re is the real part, and h is given by

$$h = \frac{1}{2}\frac{S_{0.5}+S_{-0.5}}{S_{0.5}-S_{-0.5}}. \tag{5}$$

Note that the exact estimator of [12] and zero-padded version of [17] can also be used in the estimation of δ_0. Now in order to remove the leakage terms, L_p, we propose to reconstruct and subtract them. Assume that in addition to $\hat{\delta}^{(i-1)}$, we have an estimate $\hat{A}^{(i-1)}$ of the amplitude. Then, we put $\hat{S}_p = X_p - \hat{L}_p$, where

$$\hat{L}_p = \frac{\left(\hat{A}^{(i-1)}\right)^*}{N}\frac{1+e^{-j4\pi\hat{\delta}^{(i-1)}}}{1-e^{-j\frac{2\pi}{N}(2m_0+2\hat{\delta}^{(i-1)}+p)}}. \tag{6}$$

The leakage-free coefficients, $\hat{S}_p$, can now be used in (5). Finally, the unbiased estimate of the amplitude is updated as

$$
\begin{aligned}
\hat{A}^{(i)} = \frac{1}{N}\Bigg(&\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}(m_0+\hat{\delta}^{(i)})n} \\
&- \left(\hat{A}^{(i-1)}\right)^*\frac{1-e^{-j4\pi\hat{\delta}^{(i)}}}{1-e^{-j\frac{4\pi}{N}(m_0+\hat{\delta}^{(i)})}}\Bigg).
\end{aligned} \tag{7}
$$

III. Relationship to the Algorithm of [14]

Recently, an algorithm that "filters" out the negative component and uses a windowing strategy to refine the frequency estimate has been proposed in [14]. We explain here its relationship to our approach, originally proposed in [11].

Suppose we frequency-shift the signal $x(n)$ to give $x'(n) = x(n)e^{j2\pi f_0 n}$. This places the negative frequency at dc and positive frequency at $2f_0$. Then, [14] suggests "filtering" the negative component by subtracting the signal mean, shifting the signal back by f_0, and re-estimating the frequency. Thus, [14] starts by using a windowing strategy to obtain a preliminary estimate, $\hat{f}_0$, and putting $x'(n) = x(n)e^{j2\pi\hat{f}_0 n}$. If the preliminary frequency estimate is accurate enough, then "most" of the energy of the negative component is removed.

The series of steps to shift the signal, remove the mean and then shift it back can be equivalently implemented by directly estimating the amplitude of the negative component as

$$\hat{A}_n = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{j\frac{2\pi}{N}\hat{f}_0 n}. \tag{8}$$

The "filtered" signal of [14] becomes

$$x_f(n) = x(n) - \hat{A}_n e^{-j2\pi\hat{f}_0 n}. \tag{9}$$

Therefore, similarly to our approach, the estimator of [14] essentially involves reconstructing and removing the negative component. However, there are two key differences between the two methods. First our approach does not require any windowing other than the standard rectangular window. This is important since it is well established that nonrectangular windowing suppresses the leakage at the expense of a higher estimation variance. Second, comparing (8) and (7), we see that the algorithm of [14] does not account for the effect of the leakage in the estimate of the amplitude. The biased amplitude estimates lead to biased frequency estimates and, consequently, an inferior estimation performance compared to our approach. This is confirmed in Section V, where we compare the performance of our approach with that in [14].

IV. Analysis

A. Theoretical Performance

We present asymptotic analysis of the performance of the proposed estimator. We assume that the maximum bin estimate is reliable, i.e., $\hat{m}_0 = m_0$, and put, without loss of generality, $|A| = 1$. Although the noise is assumed Gaussian, similar results can be obtained under more relaxed assumptions [8].

In what follows, we make the following assumption.

Assumption 1: For some $0 \le \alpha < 1$, the true frequency of the signal follows

$$f_0 \sim O(N^{-\alpha}).$$

Given (2), *Assumption 1* implies that $m_0 \sim O(N^{1-\alpha})$.

Let us consider the DFT coefficients at $m_0 + \delta + p$, for ($p = \pm 0.5$). Starting with S_p we have

$$S_p = \frac{b}{\delta_0 - \delta - p} + O(N^{-1}),\ b = -\frac{e^{j\phi}}{j2\pi}\left(1 + e^{j2\pi(\delta_0-\delta)}\right).$$

The leakage terms L_p and $\hat{L}_p$ are also given by

$$L_p = \frac{c}{2m + \delta_0 + \delta + p} + O(N^{-1}),\ c = \frac{e^{j\phi}}{j2\pi}\left(1 + e^{-j2\pi(\delta_0+\delta)}\right)$$

and

$$\hat{L}_p = \frac{d}{2m + 2\delta + p} + O(N^{-1}),\ d = \frac{e^{j\hat{\phi}}}{j2\pi}\left(1 + e^{-j4\pi\delta}\right).$$

We see that $S_p \sim O(1)$, whereas both L_p and $\hat{L}_p$ are $O(N^{\alpha-1})$.

Including the leakage and noise terms in the estimation determinant yields

$$\begin{aligned}\tilde{h} =& \frac{1}{2}\frac{X_{0.5} + X_{-0.5} - \hat{L}_{0.5} - \hat{L}_{-0.5} + W_{0.5} + W_{-0.5}}{X_{0.5} - X_{-0.5} - \hat{L}_{0.5} + L_{-0.5} + W_{0.5} - W_{-0.5}} \\ =& \frac{h + \lambda_+ + \gamma_+}{1 + 2\lambda_- + 2\gamma_-}.\end{aligned} \tag{10}$$

Here, we have defined the noise determinant

$$\gamma_\pm = \frac{1}{2}\frac{W_{0.5} \pm W_{-0.5}}{S_{0.5} - S_{-0.5}} \tag{11}$$

and leakage determinant

$$\lambda_\pm = \frac{1}{2}\frac{L_{0.5} - \hat{L}_{0.5} \pm (L_{-0.5} - \hat{L}_{-0.5})}{S_{0.5} - S_{-0.5}}. \tag{12}$$

As the noise is Gaussian, we have $W_{\pm 0.5} \sim O(N^{-1/2})$ and $\gamma_\pm \sim O(N^{-1/2})$, whereas $\lambda_\pm \sim O(N^{\alpha-1})$. Thus, it is evident that when $\alpha < 1/2$, the order of λ is lower than γ, and vice versa.

To find the estimation error, we define $\nu = \delta_0 - \delta$ and apply the expansion $(1+x)^{-1} = 1 - x + O(x^2)$ to (10), giving

$$\begin{aligned}\mathrm{Re}\{\tilde{h}\} =& \mathrm{Re}\{(h + \lambda_+ + \gamma_+)(1 - 2\lambda_- - 2\gamma_-)\} + O\left(N^{2\beta}\right) \\ =& \nu - \mathrm{Re}\{2\nu(\lambda_- + \gamma_-) - (\lambda_+ + \gamma_+)\} + O\left(N^{2\beta}\right)\end{aligned}$$

where

$$\beta = \max\{\alpha - 1, -1/2\}.$$

Putting $V_p = L_p - \hat{L}_p$, we have

$$V_p = \frac{c - d}{2m_0} + O\left(N^{2\beta}\right).$$

The estimation error of δ_0 is then given by

$$\begin{aligned}\epsilon_\delta =& \nu - \mathrm{Re}\{\tilde{h}\} \\ =& (\nu^2 - 0.25)\mathrm{Re}\left\{b^{-1}[(1-2\nu)(V_{0.5} + W_{0.5})\right. \\ & \left. + (1+2\nu)(V_{-0.5} + W_{-0.5})]\right\} + O\left(N^{2\beta}\right) \\ =& (\nu^2 - 0.25)\mathrm{Re}\left\{b^{-1}[(1-2\nu)W_{0.5} + (1+2\nu)W_{-0.5}\right. \\ & \left. + (c-d)m_0^{-1}]\right\} + O\left(N^{2\beta}\right).\end{aligned} \tag{13}$$

Thus, the estimation error is asymptotically normally distributed. Since $E[W_p] = 0$, the estimation bias is given by

$$\mu_\delta = \mathrm{E}[\epsilon_\delta] = \frac{\nu^2 - 0.25}{m_0}\mathrm{Re}\left\{\frac{c-d}{b}\right\} + O\left(N^{2\beta}\right). \tag{14}$$

This implies that the bias asymptotically converges to zero at the rate of $O(N^{\alpha-1})$. Finally, the asymptotic variance is

$$\sigma_\delta^2 = \mathrm{Var}[\epsilon_\delta] = \frac{\pi^2(\nu^2 - 0.25)^2(4\nu^2 + 1)}{2\rho N \cos^2(\pi\nu)} + O\left(N^{2\beta-1}\right). \tag{15}$$

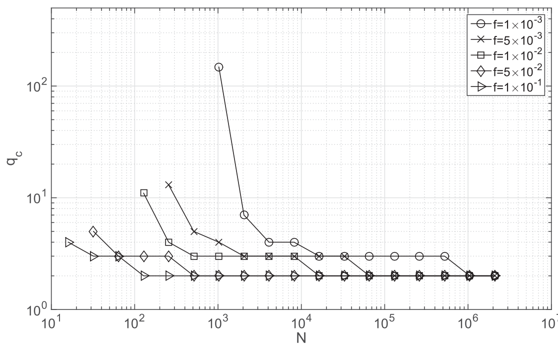

Fig. 1. Number of iterations required for convergence q_c versus N for different values of f_0.

B. Convergence

In this section, we study the convergence behavior of the proposed estimator. The determinant, $\tilde{h}(\delta)$, can be rewritten as

$$\tilde{h}(\delta) = \frac{\sin\left(\frac{2\pi}{N}(\delta_0 - \delta)\right)}{2\sin\left(\frac{\pi}{N}\right)}\left[1 + O\left(N^\beta\right)\right].$$

Expanding $\tilde{h}(\delta)$ as a Taylor series about δ_0, we have

$$\tilde{h}(\delta) = \tilde{h}(\delta_0) + (\delta - \delta_0)\tilde{h}'(\delta_0)\left[1 + O\left(N^\beta\right)\right]. \tag{16}$$

Now $\tilde{h}(\delta_0) = h(\delta_0) \sim O(N^{-1})$ and $\tilde{h}'(\delta_0)$ was shown in [8] to simplify to $\tilde{h}'(\delta_0) = -1 + O(N^{-2})$. Also, (12) implies that $\lambda_\pm(\delta_0) \sim O(N^{-2})$ and $\lambda'_+(\delta_0) \sim O(N^{\alpha-1})$. Thus,

$$\begin{aligned}\tilde{h}'(\delta_0) =& [1 + \lambda_-(\delta_0)]^{-2}\left\{\left[\tilde{h}(\delta_0) + 2\lambda_+(\delta_0))\lambda'_-(\delta_0)\right.\right. \\ & \left.\left. - (\tilde{h}'(\delta_0) + 2\lambda'_+(\delta_0)\right][1 + \lambda_-(\delta_0)]\right\} + O\left(N^\beta\right) \\ =& -1 + O(N^\beta).\end{aligned} \tag{17}$$

Substituting (17) into (16), we arrive at the following expression of the estimation function:

$$\psi(\delta) = \delta + \tilde{h}(\delta) = \left[\delta_0 + (\delta - \delta_0)O\left(N^\beta\right)\right]\left[1 + O\left(N^\beta\right)\right]. \tag{18}$$

Now, for any $\delta_1, \delta_2 \in [-0.5, 0.5]$, we have that

$$|\psi(\delta_1) - \psi(\delta_2)| = |\delta_1 - \delta_2| O\left(N^\beta\right) \tag{19}$$

which shows that $\psi(\delta)$ is a contractive mapping provided that $\beta < 0$ with a fixed point at δ_0 (that is $\psi(\delta_0) = \delta_0$). Thus, by the fixed point theorem, the proposed algorithm asymptotically converges to the fixed point at the rate of $O(N^\beta)$.

In practice, we say the algorithm has converged if the residual is of lower order than the CRB [8]. Given that the CRB is $O(N^{-1/2})$ [18], the number of iterations needed for practical convergence, q_c, is given by

$$N^{q_c\beta} < N^{-\frac{1}{2}} \Rightarrow q_c > \left\lceil -\frac{1}{2\beta} \right\rceil \tag{20}$$

where $\lceil \bullet \rceil$ returns the smallest integer greater than or equal to "$\bullet$". Equation (20) implies that when $\beta = -1/2$, in which case the noise dominates the leakage, two iterations are sufficient for convergence. On the other hand, as the frequency tends to 0, we have $\beta = \alpha - 1$ and more iterations are needed for convergence. Fig. 1 shows the required number of iterations,

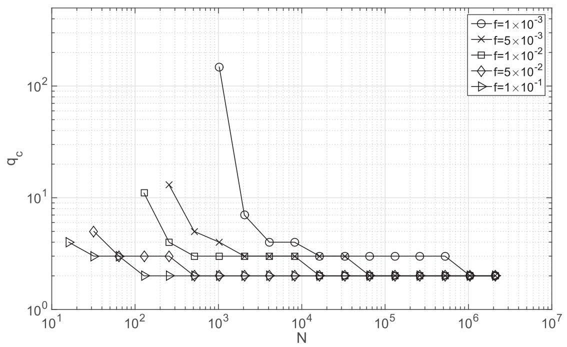

Fig. 2. RMSE of $\hat{f}_0$ versus f_0 when $N = 64$ and SNR $= 20$ dB. 5000 Monte Carlo runs are used.

q_c, versus N for different values of f_0. We see that as N or f_0 become smaller, more iterations are needed for convergence. However, as $N \to \infty$, $\beta \to -1/2$ and only two iterations are required.

Upon convergence, the asymptotic variance is obtained by substituting $\delta_0 = \delta$ in (15) yielding

$$\sigma_{\delta}^2|_{\nu=0} \approx \frac{\pi^2}{32\rho N}. \tag{21}$$

The asymptotic CRB of the frequency residual is $CRB|_{N\to\infty} = 6/(2\pi^2\rho N)$ [8]. Thus, the ratio between the asymptotic variance and asymptotic CRB is $r \approx 1.0147$.

V. Simulation Results

In this section, we illustrate the performance of the proposed estimator using simulations. For simplicity, we fix the amplitude of the test signal $a = 1$. In each Monte Carlo run, we select the phase ϕ uniformly randomly from $[-\pi, \pi]$.

First, we investigate the performance of the algorithm as a function of the frequency f_0. Due to the periodicity of the spectrum of a real signal, we show results only for $f_0 \in [0, 0.25]$. As f_0 tends to zero, the two complex components move closer to each other. On the other hand, their maximum separation occurs when f_0 shifts toward 0.25. In this test, we fix SNR $=$ 20 dB and include the curves for the CRB and the method of [14]. Fig. 2 shows the root mean square error (RMSE) of $\hat{f}_0$ versus Nf_0. Notice that the RMSE of the proposed method follows the CRB, whereas Djukanovic's method exhibits a worse performance due to the bias in the amplitude estimate.

Next, we simulate the performance versus N. In addition to the CRB and Djukanovic's method [14], we include results for the matrix pencil method (MPM) [5], WLP algorithm [6] and $\sin^{\alpha}(n)$ (Sine) windowing method [10]. In the simulation, we set $f_0 = 0.05$ and SNR $= 40$ dB. We use a pencil parameter $L = \lfloor N/3 \rfloor$, and for WLS, we employ the monic constraint and initialize the frequency estimates using the DFT. Also for the Sine Windowing technique, we set the window order $\alpha = 1$. In Fig. 3, we report the RMSE of $\hat{f}_0$ versus N. The results show that our method, WLP and MPM are very close to the CRB. However, WLP and MPM incur a significantly higher computational burden having a complexity $O(N^3)$ as opposed to our approach that has a complexity of $O(N \log_2 N)$. The figure also shows that our approach outperforms the method of [14] for small N and is significantly better than the Sine Windowing estimator for all N.

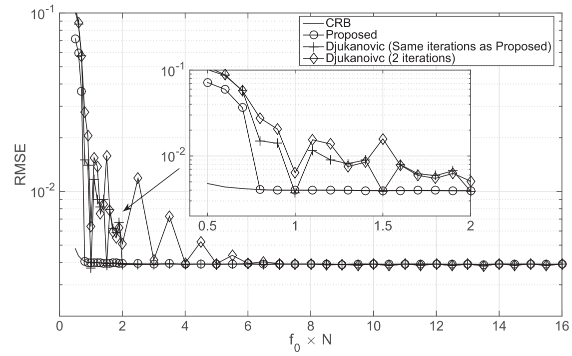

Fig. 3. RMSE of $\hat{f}_0$ versus N when $f_0 = 0.05$ and SNR $= 40$ dB. 10 000 Monte Carlo runs are used.

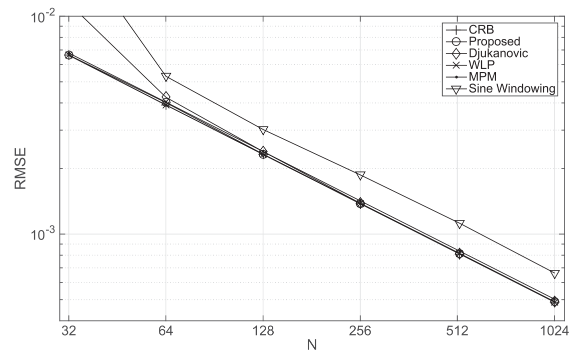

Fig. 4. RMSE of $\hat{f}_0$ versus SNR when $f_0 = 0.005$, $N = 256$. 5000 Monte Carlo runs are used.

Finally, we examine the RMSE versus SNR. Again we compare our method with MPM, WLP, Sine Windowing, and Djukanovic's method. Fig. 4 shows the results for $f_0 = 0.005$ and $N = 256$. Observe that the proposed method and WLP exhibit the best performance, having RMSE that coincides with the CRB for SNR > -2 dB (although our approach is computationally much simpler). MPM shows poorer performance at low SNR, whereas the estimator of [14] is worse both at low and high SNR. Finally, the Sine Windowing method has the worst performance for all SNRs.

VI. Conclusion

Simple DFT-based estimators for the frequency of a real sinusoidal signal in additive noise suffer from biases resulting from the interference between the negative and positive frequency components. We have, in this letter, proposed and analyzed a novel algorithm that addresses these issues effectively. The new algorithm is DFT-based and, therefore, can be efficiently implemented using the FFT algorithm. In addition to the frequency, the estimator obtains estimates of the amplitude and phase and uses them to account for the interference between the positive and negative spectral components. We demonstrated that it has a performance that practically coincides with the CRB. Although we focused on the frequency estimates, we note that the other parameters show similar behavior. Finally, we point out that the estimator is amenable to incorporating *a priori* information on the parameters, as is the case for example in power systems where the frequency is in a small region around its nominal value.

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